ELECTRON TRANSPORT ON A CRYSTAL LATTICE WITH AND WITHOUT MAGNETIC FIELD

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1. INTRODUCTION

It is classical that the the magnetic Laplacian on $L^2(\mathbb{R}^2)$;

$$-\Big(\frac{\partial}{\partial x}+\frac{ib}{2}y\Big)^2-\Big(\frac{\partial}{\partial y}-\frac{ib}{2}x\Big)^2,\quad (b\neq 0)$$

has only eigenvalues(Landau levels) $\{|b|(2k+1)\}_{k=0,1,\dots}$, where b denotes the magnetic flux. What happens when one consider its discrete analogue?

The Harper operator was designed to describe behavior of an electron moving on the square lattice exposed to a constant magnetic field by P. G. Harper [H];

$$\begin{aligned}&(H_b\varphi)(l,m)\\&=\frac{1}{4}[e^{\sqrt{-1}bm/2}\varphi(l+1,m)+e^{-\sqrt{-1}bm/2}\varphi(l-1,m)\\&\qquad+e^{-\sqrt{-1}bl/2}\varphi(l,m+1)+e^{\sqrt{-1}bl/2}\varphi(l,m-1)]\\&\qquad\qquad(\varphi\in\ell^2(\mathbb{Z}^2),(l,m)\in\mathbb{Z}^2),\end{aligned}$$

where b is the magnetic flux. Seemingly it is a very simple operator but, compared with the magnetic Laplacian on $\mathbb{R}^2$, its spectrum has a very complicated feature. For example, it was shown by M. D. Choi, G. Elliott, N. Yui [CEY] that its spectrum has a band structure (i.e a finite sum of finite closed intervals) when $b\in\mathbb{Q}$, and is a Cantor set when $b\notin\mathbb{Q}$.

There is no physical reason to distinglish a rational magnetic field from an irrational one. Thus people are interested how the spectrum of H_b depends on the magnetic flux b, and regularity of the gap edges of the spectra in b is intensively studied. It is summarized as the gap edges are Hölder continuous of 1/2-exponent, and there exisit right and left derivatives but not differentiable at $b\in\mathbb{Q}$.[HS]. See [Be2], [Be1] for more details.

J. Bellissard observed that the Harper operator is an element of a C^*-algebra, the non-commutative torus $\mathcal{A}_b$. The C^∞-structure is defined by A.Connes of the continuous field $\mathcal{A}_I =$ " $\cup_{b\in I} \mathcal{A}_b$" of the the non-commutative tori $\mathcal{A}_b$ parametrized by b in an interval I. The Harper operator is a smooth self-adjoint element of $\mathcal{A}_I$ in this sense. He showed that the gap edge of not only the Harper operator but also of a self-adjoint operator, which is smooth enough in $\mathcal{A}_I$ is Lipschiz continuous when the gap width is positive [Be1].

In this article, we discuss the same Lipschiz continuity for a magnetic transition operator on a more general graphs, namely a crystal lattice. A crystal lattice is an abelian cover of a finite graphs and typical examples are $\mathbb{Z}^d$-lattice, the triangular lattice, the hexagonal lattice.

On a crysatl lattice, we define the magnetic transition operator, which is a generalization of the Harper operator, and see it can be regarded as an element of a certain C^*-algebra. In this way, we follow the approach by J. Bellissard and obtain the similar result.

2. MAGNETIC TRANSITON OPERATORS

First of all, we recall basic properties of the magnetic Laplacian of $\mathbb{R}^d$. A magnetic field of $\mathbb{R}^d$ is a closed 2-form $B = \sum_{i<j} b_{ij} dx_i \wedge dx_j$ and a 1-form A with $dA = B$ is called a vector potential. The connection defined by A on the trivial line bundle is denoted by $\nabla_A := d - \sqrt{-1}A$ and its adjoint operator by ∇_A^*. The magnetic Laplacian Δ_A is defined by $\Delta_A = \nabla_A^* \nabla_A$. Although a vector potential for a given magnetic field B is not unique but the unitary equivalent class of Δ_A is uniquely determined for a given magnetic field. A magnetic filed B is periodic with respect to a lattice Γ in $\mathbb{R}^d$ (i.e. $\gamma^* B = B$, $\forall \gamma \in \Gamma$) if and only if an associated vector field A is weak Γ-invariant, namely there exists a function s_γ of $\mathbb{R}^d$ such that

$$\gamma^* A - A = ds_\gamma \quad (\forall \gamma \in \Gamma).$$

We take this as a model and define magnetic transition operators on a crystal lattice as its discretizations.

Let us denote the set of all oriented edges of X by E, the origin and the terminus of $e \in E$ by $o(e)$ and $t(e)$, respectively, the inverse edge of e by $\overline{e}$. A positive function $p : E \to \mathbb{R}_+$of E is called a *symmetric transition probability* when it admits a positive function $m : X \to \mathbb{R}_+$ of X satisfying

$$\sum_{e\in E_x} p(e) = 1 \quad (\forall x \in X),$$

$$m(o(e))p(e) = m(t(e))p(\overline{e}) =: m(e) \qquad (\forall e \in E),$$

where $E_x = \{e \in E \mid o(e) = x\}$. A function $\omega : E \to \mathbb{R}$ satisfies

$$\omega(\overline{e}) = -\omega(e),$$
$$\gamma^*\omega - \omega = ds_\gamma \quad (\forall \gamma \in \Gamma)$$

is said to be a weak Γ-invariant 1-form. It is a discrete analogue of a vector potential.

Put

$$\ell^2(X) := \{\varphi : X \to \mathbb{C} \mid \|\varphi\|^2 = \sum_{x \in X} m(x)|\varphi(x)|^2\}.$$

Given a Γ-invariant transition probability p and a weak Γ-invariant 1-form ω, we define the *magnetic transition operator* $H_\omega : \ell^2(X) \to \ell^2(X)$ by

$$H_\omega \varphi(x) := \sum_{e \in E_x} p(e) e^{-\sqrt{-1}\omega(e)} \varphi(t(e)).$$

Note that it is the transition operator for a symmetric random walk when $\omega \equiv 0$.

Example: Restrict the weak $\mathbb{Z}^2$-invariant 1-form $A = \frac{b}{2}(-ydx + xdy)$ of $\mathbb{R}^2$ to the $\mathbb{Z}^2$-lattice, we have a weak $\mathbb{Z}^2$-invariant 1-form ω of the $\mathbb{Z}^2$-lattice. The magnetic transition operator associated with this ω and the transition probability p with $p(e) = 1/4$ for all edges e of the $\mathbb{Z}^2$-lattice is nothing but the Harper operator.

We say "magnetic" transition operators but what is the "magnetic field"? As a crystal lattice X is a one-dimensional object, we cannot define a magnetic field as a closed 2-form but we use the group cohomology instead. For that, recall the exact sequence:

$$0 \to H^1(\Gamma, \mathbb{R}) \overset{\iota}{\to} H^1(X_0, \mathbb{R}) \overset{\pi^*}{\to} H^1(X, \mathbb{R})^\Gamma \overset{\Theta}{\to} H^2(\Gamma, \mathbb{R}) \to 0.$$

A weak Γ-invariant 1-form ω defines a class of Γ-invariant cohomology in $H^1(X, \mathbb{R})^\Gamma$. The magnetic transiton operators associated with weak Γ-invariant 1-form of the same class in $H^1(X, \mathbb{R})^\Gamma$ are unitarily equivalent. We call $\Theta[\omega] \in H^2(\Gamma, \mathbb{R})$ *magnetic flux class* of $[\omega]$.

When X is the universal abelian cover of X_0, i.e. $\Gamma = H_1(X_0)$, in the above exact sequence, we have $H^1(\Gamma, \mathbb{R}) = \mathrm{Hom}(\Gamma, \mathbb{R}) = H^1(X_0, \mathbb{R})$ and ι is isomorphic. Thus so it Θ. In this case, for $B \in H^2(\Gamma, \mathbb{R})$, there exists the unique class $[\omega] \in H^1(X, \mathbb{R})^\Gamma$ of weak Γ-invariant 1-form ω such that $\Theta[\omega] = B$. All H_ω's are unitarily equivalent and we sometimes denote them even by H_B when we are concerned with their spectra. For a general abelian cover X, $\Theta[\omega_1] = \Theta[\omega_2]$ only means that $\omega_2 = \omega_1 + \pi^*\omega_0$ and there remains some freedom for the choice of bounded 1-forms $\omega_0 \in H^1(X_0, \mathbb{R})/H^1(\Gamma, \mathbb{R})$. We remark that there is, however, the standard representative ω_B for a given B, with which the *central limit theorem* holds [K1].

3. Formulation by using the C^*-algebras

Without magnetic field, the (magnetic) transitin operator H_0 commutes with the Γ-action, which allows us to use the direct integral decomposition of the regular representation of Γ to analyze the spectrum of H_0. When a magnetic field presents, the magnetic transition operator no more commutes with the Γ-action but it does with the magnetic translations. To make use of this, J. Bellissard associated the Harper operator H_b with an element of the non-commutative torus $\mathcal{A}_b$ defined by M. A. Rieffel[Rie]. Then by using the notion of the continuous field of C^*-algebras: $b \in I \mapsto \mathcal{A}_b$, he studied the continuity of the spectra of H_b in the magnetic flux b[Be1]. In the similar way, our magnetic transition operators can be considered as elements of a certain C^*-algebra [S].

Let $X \xrightarrow{\Gamma} X_0$ be a crystal lattice. and take a weak Γ-invariant 1-form ω:

$$\gamma^*\omega - \omega = ds_\gamma \quad (\forall \gamma \in \Gamma)$$

associated with $\Theta(\omega) = B \in H^2(\Gamma, \mathbb{R})$. We denote the unitary equivalent class of H_ω by H_B. Put $W = C(X_0, \mathbb{C})$, the space of functions of the finite graph X_0. Note it is a finite dimensional Hilbert space. Let $\ell^2(\Gamma, W)$ be the space of W-valued ℓ^2 functions of Γ and define the right magnetic translations $U_\alpha : \ell^2(\Gamma, W) \to \ell^2(\Gamma, W)$, for $\alpha \in \Gamma$, by

$$(U_\alpha \phi)(\gamma) = e^{sqrt{-1}B(\gamma,\alpha)}\phi(\gamma\alpha) \quad (\phi \in \ell^2(\Gamma, W)).$$

It is easy to check $U_\alpha U_\beta = e^{\sqrt{-1}B(\alpha,\beta)}U_{\alpha\beta}$, $U_\alpha^* = U_\alpha^{-1}$. We give

$$C(\Gamma, B) := \{A = \sum_{\alpha:\text{finite sum}} a(\alpha)U_\alpha : a(\alpha) \in \mathrm{End}(W)\}$$

a $*$-algebra structure by

$$\Big(\sum a_\alpha U_\alpha\Big) \cdot \Big(\sum b_\beta U_\beta\Big) = \sum e^{\sqrt{-1}B(\alpha,\beta)} a_\alpha b_\beta U_{\alpha\beta}$$

$$*\Big(\sum a_\alpha U_\alpha\Big) = \sum a^*_{\alpha^{-1}} e^{-\sqrt{-1}B(\alpha,\alpha^{-1})} U_\alpha$$

and the completion of $C(\Gamma, B) \subset \mathbb{B}(\ell^2(\Gamma, W))$ with respect to the operator norm is denoted by $\mathcal{A}(\Gamma, B)$. It is the C^*-algebra we are going to use.

The unitary equivalence between $\ell^2(\Gamma, W)$ and $\ell^2(X)$ is given, by defining $\phi \in \ell^2(\Gamma, W)$, for $\varphi \in \ell^2(X)$, by

$$\phi(\alpha)(x_0) = \varphi(\alpha x_0)e^{\sqrt{-1}s_\alpha(\alpha x_0)} \quad (\alpha \in \Gamma, x_0 \in X_0).$$

Through this identification, we regard H_B as an element of $C(\Gamma, B)$, given explicitely as

$$H_B = \sum a_\alpha U_\alpha,$$

where $a_\alpha \in \mathrm{End}(W)$ is defined by

$$(a_\alpha \psi)(x_0) = \sum_{e \in E_{x_0}, \gamma^{-1} t(e) \in X_0} p(e) e^{-\sqrt{-1}(\omega(e) + s_\gamma(t(e)))} \psi(\gamma^{-1} t(e)).$$

Now take a Γ-equivariant map $\Phi : X \to \Gamma \otimes \mathbb{R}$ and consider B as a skewsymmetric 2-form of $\Gamma \otimes \mathbb{R}$, then a representative ω of $\Theta[\omega] = B$

$$\omega(e) = B(d\Phi(e), \Phi(o(e)),$$
$$s_\alpha(x) = B(\alpha, \Phi(x))$$

gives a smooth a_α in B.

4. Differential structures

In the previous section, we see $H_B \in C(\Gamma, B) \subset \mathcal{A}(\Gamma, B)$. As we want to study how the spectra of H_B depend on B, we need a notion of a "collection" "$\cup_B \mathcal{A}(\Gamma, B)$" of all these C^*-algebras in which each of H_B belongs to.

Let B be a magnetic flux class and Ω be its open neighborhood in $H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}^{d(d-1)/2}$. Put $\theta : B \in \Omega \mapsto e^{\sqrt{-1}B(\cdot,\cdot)} \in H^2(\Gamma, \mathbb{R})$ and U_α^Ω are the formal unitary elements satisfying

$$U_\alpha^\Omega U_\beta^\Omega = \theta(\alpha, \beta) U_{\alpha\beta}^\Omega,$$
$$a_\alpha U_\beta^\Omega = U_\beta^\Omega a_\beta,$$
$$(U_\alpha^\Omega)^* = (U_\alpha^\Omega)^{-1} = U_{\alpha^{-1}}^\Omega.$$

Put

$$\mathcal{P}_\Omega^k = \{A = \sum a_\alpha U_\alpha^\Omega \text{ finite sum} \mid a_\alpha \in C^k(\Omega, \mathrm{End}(W))\},$$

and define the evaluation map $\rho_B : \mathcal{P}_\Omega^k \to \mathcal{P}_B^k$ by $U_\alpha^\Omega \mapsto U_\alpha^B$. The C^*-norm of $\mathcal{P}_B^k$ is given as usual by

$$\|A\|_B = \sup_{\pi \in \mathrm{Rep}} \|\pi(A)\|,$$

and let $\mathcal{A}_\Omega$ be the completion of $\mathcal{P}_\Omega^k$ with respect to

$$\|A\|_\Omega = \sup_{B \in \Omega} \|\rho_B(A)\|_B.$$

We also define the trace $\tau : \mathcal{A}_\Omega \to C(\Omega)$ by $\tau(A) = \dim(W)^{-1} \mathrm{tr}_W\, a_0$.

The differencial structure of $\mathcal{A}_\Omega$ is endowed, for $A \in \mathcal{P}_\Omega^\infty$,

$$\partial_i A = \sum_{\alpha\in\Gamma} \sqrt{-1}\alpha_i a_\alpha U_\alpha \quad (i = 1, \ldots, d),$$

$$\delta_{ij} A = \sum_{\alpha\in\Gamma} \frac{\partial a_\alpha}{\partial b_{ij}} U_\alpha, \quad (1 \le i < j \le d).$$

Here ∂ represents a derivation in space and δ that in the magnetic flux. We see that ∂_i is a $*$-derivation and

$$\delta_{ij}(*A) = *\delta_{ij}(A)$$

$$\delta_{ij}(AB) = (\delta_{ij}A)B + A(\delta_{ij}B) - \sqrt{-1}(\partial_i A \partial_j B - \partial_j A \partial_i B).$$

Thus it is reasonable to define the order of each is $\mathrm{ord}(\partial_i) = 1$, and $\mathrm{ord}(\delta_{ij}) = 2$. The space

$$\mathcal{C}^{l,n}(\mathcal{A}_\Omega) = \{A \in \mathcal{A}_\Omega \mid \|\delta^s \partial^r(A)\| < \infty, 0 \le |s| \le l, 0 \le 2|s| + |r| \le n\},$$

is dense in $\mathcal{A}_\Omega$ and there exist a norm which makes it a Banach algebra.

5. LIPSCHTIZ CONTINUITY

Note that we don't use the fact X being an abelian cover(i.e. $\Gamma \cong \mathbb{Z}^d$) so far. Indeed, everything works well with a Γ-cover X of a finite graph X_0 where Γ is not necessarily abelian. Finally, from now on, the assumption $\Gamma \cong \mathbb{Z}^d$ is essential.

Let Ω be an open neighborhood of B_0 in $H^2(\Gamma, \mathbb{R})$ and we put $B' = B_0 + hB \in \Omega$. $(h \in (0, \epsilon))$. As we consider B as a skew-symmetric 2-form of $\Gamma\otimes\mathbb{R}$, we take the orthogonal complement V of the null space of B in $\Gamma\otimes\mathbb{R}$ and write the orthogonal projection to V by $p : \Gamma\otimes\mathbb{R} \to V$. The Weyl representation $(\pi_h, \mathcal{H}_w)$ of $\mathcal{A}_{hB}$ on $\mathcal{H}_w \subset L^2(V, W)$ is given by

$$(\pi_h(U_\alpha^B)\varphi)(x) = e^{\sqrt{-1}B(x,\sqrt{h}\alpha)}\varphi(x + \sqrt{h}p(\alpha)) \quad (\alpha \in \Gamma, \varphi \in \mathcal{H}_w).$$

We also define a $*$-homomorphism $\mathcal{A}_{B'} \to \mathcal{A}_{B_0} \otimes \mathbb{B}(\mathcal{H}_w)$ by $U_\alpha^{B'} \to U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B)$ and let $\mathcal{A}'$ be the C^* subalgebra of $\mathcal{A}_{B_0} \otimes \mathbb{B}(\mathcal{H}_w)$ generated by $U_\alpha^{B_0} \otimes \pi_h(U_\alpha^B)$. Then it is isomorphic to $\mathcal{A}_{B'}$. We express $A = \sum a_\alpha U_\alpha \in \mathcal{A}_{B'}$ as an element of $\mathcal{A}'$ through this isomorphism; Let

$$a(h, \xi) := \sum a_\alpha(B') e^{\sqrt{-1}B(\alpha,\xi)} e^{h|p(\alpha)|_B^2/2} U_\alpha^{B_0} \in \mathcal{A}_{B_0},$$

and $T_\xi : \mathcal{H}_w \to \mathcal{H}_w$ be the integral operator with its kernel

$$t_\xi(x, y) = \pi^{-k} e^{\sqrt{-1}B(\xi, y-x)/2} e^{-|y-x|_B^2/4},$$

then

$$A(B') = (4\pi h)^{-k} \int_V a(h, \xi) \otimes T_{\frac{\xi}{\sqrt{h}}} d_B\xi.$$

By using this expression, we treat all $A(B')$ together and reduce the problem to estimate of elements $a(h,\xi)$ in $\mathcal{A}_{B_0}$ and obtain the following theorem.

Theorem 1. *Let $H \in \mathcal{C}^{1,d/2+2+\epsilon}(\mathcal{A}_\Omega)$ be a self-adjoint element and denote its gap edges by $E(B)$. If the gap width is positive at $B_0 \in \Omega$, then in a small neighborhood $U(B_0)$ of B_0, $E(B)$ is a Lipschitz function, namely, for $B_1, B_2 \in U(B_0)$,*

$$|E(B_2) - E(B_1)| \leq c(H)[\sup_{B \in U(B_0)} W^g(B)]^{-(d/2+4)}|B_2 - B_1|$$

holds.

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