Renormalization group approach to a generalization of the law of iterated logarithms for one-dimensional (non-Markovian) stochastic chains

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1 Introduction.

Renormalization group (RG) is, roughly speaking, a dynamical system determined by a map which represents a response of (a set of random) objects in consideration to a change of accuracy of observation, or 'scale transformation', on a parameter space of generating functions of quantities defined on the objects. The method is expected to analyze the asymptotic behaviors and critical phenomena of random objects.

We can think of, and there have been deep works on, various objects, for which the RG approach may be effective. For the purpose of exhibiting the RG idea, we shall here focus on a simplest object for which the RG method is non-trivial, a class of probability measures on a set of paths (stochastic chains) on $\mathbb{Z}$. The RG approach focuses on (stochastic and/or approximate) similarity of the object (paths, in our case), rather than on Markov and/or martingale properties, hence the RG has a (yet to be explored) possibility of being a complimentary tool to these well-established methods.

One other point about introducing RG approaches to stochastic chains is that, like differential equations and stochastic differential equations, RG can be seen as a differential type equation which determine the object (stochastic chain, in our case) as a solution to a RG equation. In fact, we will see that, given an arbitrary one dimensional RG, we can uniquely construct a stochastic chain consistent with the equation. The RG approach to the simple random walk on $\mathbb{Z}$ has been known in mathematics [7]. Our standpoint is to place RG in the center, instead of regarding RG as another method of constructing well-known stochastic processes, and to show that there is a large class of stochastic chains, including simple random walks and self-avoiding paths, for which RG acts naturally, and to show, in particular, that a generalization of the law of iterated logarithms hold for such chains.

2 Definitions and main results.

2.1 Generating function and the renormalization group.

By a path (on $\mathbb{Z}$), we mean a sequence of integers such that each neighboring pair in the sequence differs by 1. Namely, a sequence $w = (w(0), w(1), \cdots, w(L(w)))$ of $\mathbb{Z}$ is a path if

$$|w(i) - w(i+1)| = 1, \quad i = 0, 1, 2, \cdots, L(w), \tag{1}$$

where $L(w)$ is the length (number of steps) of w. We write $L(w) = \infty$ for a infinite sequence (a path of infinite length). We are interested on the long distance asymptotic behaviors of paths on $\mathbb{Z}$.

Denote by Ω_1 a set of finite path with $w(0) = 0$ and $w(L(w)) = 2$, which do not hit ± 2 before the final step. Namely, a path in Ω starts at 0, jumps among ± 1 and 0, and finnally stops at 2. Define the generating function of path length L for Ω_1 by

$$\Phi_1(z) = \sum_{w \in \Omega_1} b_1(w) z^{L(w)}, \tag{2}$$

where the weights $b_1(w)$ are arbitrary non-negative constants such that the right hand side of (4) has a non-zero radius of convergence.

For $n = 1, 2, 3, \cdots$ define the functions $\Phi_n : \mathbb{C} \to \mathbb{C}$ recursively by

$$\Phi_{n+1}(z) = \Phi_1(\Phi_n(z)), \; n = 1, 2, \cdots, \tag{3}$$

and, extending the definition of Ω_1, we denote by Ω_n a set of finite path with $w(0) = 0$ and $w(L(w)) = 2^n$, which do not hit $\pm 2^n$ before the final step. The following fact is the starting point of everything.

Proposition 1 (RG on the paths on $\mathbb{Z}$) *For each $n \in \mathbb{Z}_+$, there exists a weight function $b_n : \Omega_n \to \mathbb{R}_+$ such that*

$$\Phi_n(z) = \sum_{w \in \Omega_n} b_n(w) z^{L(w)}, \tag{4}$$

holds for z within the raidius of convergence. $\diamond$

Prop. 1 follows from a decimation procedure, 'a scale transformation of accuracy of observation', and is the core of relating the renormalizationg group, the dynamical system determined by Φ_1 through (3), to path spaces Ω_n, $n \in \mathbb{Z}_+$. The definition of decimation procedure, or the definition of $b_n(w)$ is esential in proving the results in the following, but since it takes a while to understand the definition, we will leave the precise definition to a textbook [1, §5], and be content with reproducing a figure which suggests how the decimation procedure of relates a path of a small scale to a path of a large scale. We also

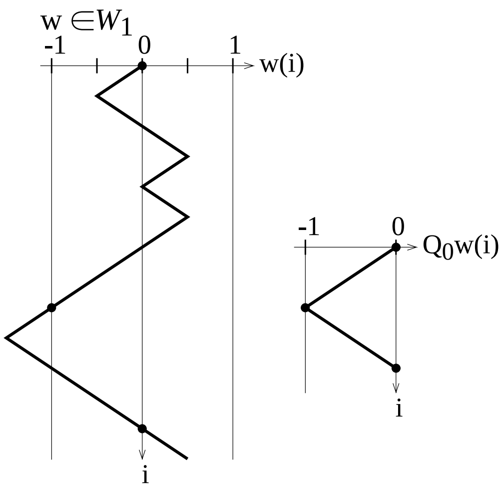

Fig. 1:

note that the simple random walk on $\mathbb{Z}$ corresponds to the case

$$b_n(w) = 1, w \in \Omega_n\,,\ \ n \in \mathbb{N}.$$

Next we define a probability measure on Ω_n, induced naturally by Φ_n, the generating function of path length. Let x_c be a positive fixed point of $\Phi_1(x)$;

$$\Phi_1(x_c) = x_c\,,\quad x_c > 0. \tag{5}$$

Then (3) implies $\Phi_n(x_c) = x_c\,,\ n \in \mathbb{N}$. Consider a probability measure determined, for each n, by

$$\mathrm{P}_n[\ \{w\}\ ] = b_n(w){x_c}^{L(w)-1},\quad w \in \Omega_n\,. \tag{6}$$

The Laplace transform of distribution of path length L with respect to the path measure P_n is calculated from (6) and (4), to obtain

$$\sum_{k\in\mathbb{Z}_+} e^{-t\,k}\mathrm{P}_n[\ \{w \in \Omega_n \mid L(w) = k\}\ ] = \frac{1}{x_c}\Phi_n(e^{-t}x_c). \tag{7}$$

$n \to \infty$ and $L \to \infty$ are related through Tauberian type theorems. In considering asymptotic behaviors, it is natural to normalize L by a scaling factor corresponding to the average growth of L in n. We will see that the appropriate scaling factor is λ^n, where

$$\lambda = \Phi_1'(x_c) = \frac{d\,\Phi_1}{dx}(x_c). \tag{8}$$

Denote by $\tilde{\mathrm{P}}_n$, the distribution of $\lambda^{-n}L$ under P_n;

$$\tilde{\mathrm{P}}_n[\ \{\lambda^{-n}k\}\ ] = \mathrm{P}_n[\ \{w \in \Omega_n \mid L(w) = k\}\ ],\quad k \in \mathbb{Z}_+, . \tag{9}$$

Substituting $t = s\lambda^{-n}$ in (7), we find $\displaystyle\sum_{\xi\in\lambda^{-n}\mathbb{Z}_+} e^{-s\,\xi}\tilde{\mathrm{P}}_n[\ \{\xi\}\ ] = \frac{1}{x_c}\Phi_n(e^{-\lambda^{-n}s}x_c)$. We will see that this quantity converges as $n \to \infty$ (Thm. 4). This means that Prop. 1 implies asymptotic behaviors of length distribution of paths.

We shall call the dynamical system on $\mathbb{R}_+$ determined by the recursion equation (3), the **renormalization group (RG)** of the sequence of probability measures on paths determined by (6).

2.2 Analysis of RG and asymptotic distribution of path length.

Write (2) as

$$\Phi_1(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{10}$$

where $c_k = \displaystyle\sum_{w\in\Omega_1;\ L(w)=k} b_1(w)$, $k \in \mathbb{Z}_+$. Hereafter we assume the following

Condition 1:

(i) The radius of convergence r of (10) is positive,

(ii) $c_0 = c_1 = 0$,

(iii) $c_2 > 0$,

(iv) $c_k \geqq 0$, $k = 3, 4, 5, \cdots$,

(v) $c_k > 0$ for some $k \geqq 3$. $\diamond$

We note that Condition 1 imposes only mild conditions on $\{b_1(w)\}$. Thus we have a rich class of stochastic chains for which the following results are applicable. We will give explicit examples in §3.

The following is elementary.

Proposition 2 *The following hold.*

(i) There exists a unique x_c (positive fixed point) such that $\Phi_1(x_c) = x_c$, $0 < x_c < r$.

(ii) $\lambda = \Phi_1'(x_c) > 2$. $\diamond$

Prop. 2 further leads to the following, also elementary, facts [1, §5].

Theorem 3 *(i) $\Phi_n(x_c) = x_c$ and $\Phi_n'(x_c) = \lambda^n$, for $n = 1, 2, 3, \cdots$.*

(ii) For all x satisfying $0 \leqq x < x_c$, $\lim_{n\to\infty} \Phi_n(x) = 0$, and $\overline{\lim_{n\to\infty}}\, 2^{-n} \log \Phi_n(x) < 0$. $\diamond$

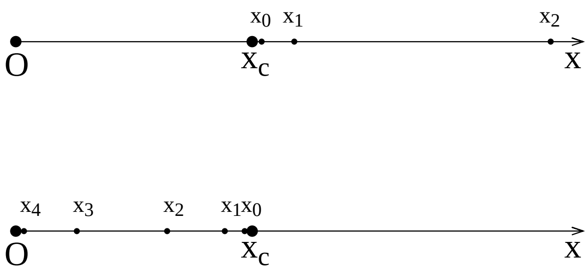

Fig. 2:

For $x > 0$, let us write for simplicity, $x_0 = x$ and $x_n = \Phi_n(x)$, $n \in \mathbb{Z}_+$. We looked into the trajectories of RG (behavior of sequences $\{x_n\}$ with different x's). Thm. 3 says that x_c is a unstable fixed point of Φ_1, and that if $0 \leqq x < x_c$, x_n converges to 0 as fast as e^{-c2^n}. It is also easy to prove that if $x > x_c$, x_n diverges (Fig. 2).

Theorem 4 *Assume that a sequence of functions $\Phi_n : \mathbb{C} \to \mathbb{C}$, $n = 1, 2, 3, \cdots$, satisfies Condition 1 at the beginning of §2.2. Put $G_n(s) = \dfrac{1}{x_c}\Phi_n(e^{-\lambda^{-n}s}x_c)$, $n = 1, 2, 3, \cdots$, for those s such that the right hand side is analytic, where λ is as in Prop. 2. Then the following hold.*

(i) G_n is defined on $\mathbf{Re}(s) \geqq 0$ and there exists a Borel probabiity measure $\tilde{\mathrm{P}}_n$ satisfying

$$G_n(s) = \int_0^\infty e^{-s\xi} \tilde{\mathrm{P}}_n[\, d\xi\,], \quad \mathbf{Re}(s) \geqq 0.$$

Furthermore, $\tilde{\mathrm{P}}_n$ *converges, as* $n \to \infty$*, to a Borel probability measure* $\tilde{\mathrm{P}}_*$ *supported on* $\mathbb{R}$*. The generating function* $G^*(s) = \int_0^\infty e^{-s\xi}\tilde{\mathrm{P}}_*[\,d\xi\,]$ *of* $\tilde{\mathrm{P}}_*$ *is defined and analytic on* $\mathbf{Re}(s) \geqq 0$ *and also on* $|s| < C_\infty$ *for some* $C_\infty > 0$*.* $G_n(s)$ *converges as* $n \to \infty$ *to* $G^*(s)$ *uniformly on any bounded closed set in* $|s| < C_\infty$*.*

(ii) There exist positive constants C *and* C'*, such that for any sequence* $\{\alpha_n\}$ *with positive elements satisfying* $\lim_{n\to\infty} 2^{n(1-\nu)/\nu}\alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$*, it holds that*

$$-C \leqq \varliminf_{n\to\infty} {\alpha_n}^{\nu/(1-\nu)} \log \tilde{\mathrm{P}}_n[\,[0,\alpha_n]\,] \leqq \varlimsup_{n\to\infty} {\alpha_n}^{\nu/(1-\nu)} \log \tilde{\mathrm{P}}_n[\,[0,\alpha_n]\,] \leqq -C'\,, \qquad (11)$$

and

$$-C \leqq \varliminf_{x\to 0} x^{\nu/(1-\nu)} \log \mathrm{P}_*[\,[0,x]\,] \leqq \varlimsup_{x\to 0} x^{\nu/(1-\nu)} \log \mathrm{P}_*[\,[0,x]\,] \leqq -C', \quad x > 0\,. \qquad (12)$$

Here we defined

$$\nu = \frac{\log 2}{\log \lambda}\,. \qquad (13)$$

Furthermore, there exist positive constants (independent of ξ *and* n*)* C*,* C'*,* C'' *such that for any* ξ *and* n *satisfying* $\left(\frac{\lambda}{2}\right)^n \xi \geqq C''$*,*

$$\tilde{\mathrm{P}}_n[\,[0,\xi]\,] \leqq C' e^{-C\xi^{-\nu/(1-\nu)}} \qquad (14)$$

holds.

(iii) $\tilde{\mathrm{P}}_*$ *has a* C^∞ *density function* ρ *with respect to the Lebesgue measure;* $\tilde{\mathrm{P}}_*[\,d\xi\,] = \rho(\xi)\,d\xi$*.* ρ *satisfies* $\rho(\xi) = 0$*,* $\xi < 0$*, and* $\rho(\xi) > 0$*,* $\xi > 0$*.*

(iv) There exists a positive constant b_0 *satisfying the following. For* $b > b_0$ *and* $n \in \mathbb{N}$*, if we put*

$$g_n(\xi) = \frac{1}{\sqrt{2\pi}h_n} e^{-\xi^2/(2h_n^2)}, \ \xi \in \mathbb{R}, \quad h_n = b\,\lambda^{-n}\sqrt{n},$$

then $\lim_{n\to\infty} \int_{\mathbb{R}} g_n(\xi-\eta)\tilde{\mathrm{P}}_n[\,d\eta\,] = \rho(\xi)$*, uniformly in* $\xi \in \mathbb{R}$*.* $\diamond$

See [1, §5] for a proof. Note that the arguments in §2.2 are based solely on (3), the RG for Φ_n defined by (10), and is independent of path measures.

2.3 Construction of the stochastic chain associated to RG.

To obtain a stochastic chain associated to P_n on Ω_n, we need to note that a stochastic chain by definition requires a distribution of points at each fixed time, while P_n is a measure on a set of paths with fixed endpoints.

Theorem 5 *Let* (Ω_n, P_n)*,* $n \in \mathbb{N}$*, be a sequence of probability spaces defined by (4) (5) (6). Then there exists a stochastic chain on* $\mathbb{Z}$*,* $W_0, W_1, W_2, \cdots$*, such that for all* $w \in \Omega_n$*,* $\mathrm{P}[\,W_j = w(j),\ j = 0,1,2,\cdots,L(w)\,] = \frac{1}{2}\mathrm{P}_n[\,\{w\}\,]$*.* $\diamond$

A main ingredient of a proof of Thm. 5 is the Kolmogorov extension theorem [1, §C].

2.4 Generalized law of iterated logarithms.

One of the consequence of RG analysis in §2.2 on the corresponding stochastic chain constructed in §2.3 is the following.

Theorem 6 (Generalized law of iterated logarithms) *W_k, $k \in \mathbb{Z}_+$, as above, satisfies the following generalized law of iterated logarithms; namely, there exists $C_\pm > 0$ such that*

$$C_- \leqq \overline{\lim_{k\to\infty}} \frac{|W_k|}{\psi(k)} \leqq C_+\,, \ \text{a.e.}.$$

Here we wrote $\psi(k) = k^\nu (\log\log k)^{1-\nu}$. The constant ν in the exponent of ψ is given by (13). $\diamond$

Note that Prop. 2 implies $0 < \nu < 1$.

The original law of iterated logarithms is known to hold for a large class of Markov processes (see, for example, [3, §VIII.5]), where in the proof of the lower bound, Markov property is essentially used. The stochastic chain constructed in §2.3 lacks Markov property in general. The generalized law Thm. 6 is applicable to cases where existing methods and results do not apply.

Idea of a proof of the upper bound of generalized law of iterated logarithms is as follows. For $x \in \mathbb{N}$, let $n = n(x)$ be defined by

$$2^{n(x)+1} > x \geqq 2^{n(x)}. \tag{15}$$

By considering hitting times of $\pm 2^n$, Thm. 5, and (14) in Thm. 4(ii), we have

$$\tilde{\mathrm{P}}_n[\ [0, \lambda^{-n}k]\ ] \leqq C' e^{-C(\lambda^{-n}k)^{-\nu/(1-\nu)}}$$

for all k and n satisfying $2^{-n}k \geqq C''$, where C, C', and C'' are some constants independent of k and n. See [1, §5] for details of the argument. This with (15) and a Borel-Cantelli type argument [1, §2.3] implies $\overline{\lim_{k\to\infty}} \dfrac{|W_k|}{\psi(k)} \leqq C^{-(1-\nu)}$, a.e..

A proof of the lower bound of the generalized law of iterated logarithms is more involved. Considering hitting times of $\pm 2^n$, and then Thm. 4, Thm. 5, and Thm. 4(ii), are used, with an argument in [4, 5] and a theorem of 2nd Borel-Cantelli type [1, §5], to find $\mathrm{P}[\ \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{\lambda^{-m} T_m (\log m)^{(1-\nu)/\nu} \leqq (C+\epsilon)^{(1-\nu)/\nu}\}\ ] = 1$, which, through a standard argument [1, §5] implies the lower bound.

3 Example: Self-repelling walk on $\mathbb{Z}$.

As examples of stochastic chains for which our results can be applied, we explain a class of chains which we call self-repelling walks [5]. The class continuously interpolates the simple random walk and the self-avoiding walk on $\mathbb{Z}$ in terms of the exponent ν. This example shows that our framework based on RG ideas contains something which were unnoticed in the previous approaches.

It is not difficult to see from the definitions that Φ_1 of the RG for the simple random walk on $\mathbb{Z}$ and the self-avoiding walk on $\mathbb{Z}$ are, respectively [1, §5],

$$\Phi_1(z) = \begin{cases} \dfrac{z^2}{1-2z^2}\,, & \text{simple random walk,} \\ z^2, & \text{self-avoiding walk.} \end{cases}$$

A simplest interpolation would be, obviously, to define a class of Φ_1 parametrized by $u \in [0,1]$, by

$$\Phi_{1,u}(z) = \frac{z^2}{1-2u^2z^2}, \quad |z| < \frac{1}{u\sqrt{2}}\,, \tag{16}$$

and define Φ_n, $n = 2,3,4,\cdots$, inductively by $\Phi_{n+1,u}(z) = \Phi_{1,u}(\Phi_{n,u}(z))$, $n \in \mathbb{Z}_+$. The case $u = 1$ corresponds to the simple random walk, and the case $u = 0$ to the self-avoiding walk. (The self-avoiding walk on $\mathbb{Z}$ is just a deterministic, straight going motion.) For any $u \in (0,1]$, $\Phi_1 = \Phi_{1,u}$ satisfies the Condition 1 at the beginning of §2.2, hence all the results of the previous sections hold. The exponent $\nu = \nu_u$ which determines the asymptotic properties, such as the generalized law of iterated logarithms Thm. 6 and the displacement exponent Thm. 7, of the corresponding stochastic chain $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, is

$$x_{c,u} = \frac{1}{4u^2}(\sqrt{1+8u^2}-1), \quad \lambda_u = \frac{\partial \Phi_{1,u}}{\partial x}(x_{c,u}) = \frac{2}{x_{u,c}} = \sqrt{1+8u^2}+1, \quad \nu_u = \frac{\log 2}{\log \lambda_u}\,. \tag{17}$$

In particular ν_u is continuous in u. Namely, the class of stochastic chains $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, $0 \leqq u \leqq 1$, continously interpolates the self-avoiding walk ($\nu = \nu_0 = 1$) and the simple random walk ($\nu_1 = 1/2$). Such continous interpolation has not been known. The RG picture gives such interpolation in a most natural way. Comparing with (4) and (16), $\{b_1(w)\}$ also can be obtained. Its explicit form, however, is not simple [5]. The obtained chains lack Markov properties, in general. The RG method works without Markov properties. Since all the results in the previous sections hold for the self-repelling walks, the generalized law of iterated logarithms Thm. 6 also hold.

The simple random walk allows 'free' motion of the paths, while in the self-avoiding walk, returning to previously visited points are strictly forbidden. In this sense, we call the obtained class of stochastic chain $W_{u,k}$, $k \in \mathbb{Z}_+$, $0 \leqq u \leqq 1$, self-repelling walks on $\mathbb{Z}$. The parameter u can be extrapolated to $u > 1$ and all the results in the previous sections hold. Naturally, we expect the resulting chain to be self-attractive. A self-repelling walk has a continuum (scaling) limit (self-repelling process) [4], a continuous time non-trivial stochastic process. Detailed properties are known [4]. In fact, some estimates are slightly easier, becasue of self-similarity, hence the fixed endpoint self-repelling process has been known [4] before the stochastic chain [5].

Another typical asymptotic property, the displacement exponent deals with expectations. An upper bound for $\mathrm{E}[\,|W_k|^s\,]$ has similar implication to that for the law of iterated logarithms, in that, a typical path moves back and forth, thus it cannot go much far compared to its path length. In fact, the upper bound is proved in the general framework of the previous sections for all the chains constructed in §2.3 [1, §5]. A lower bound, on the other hand, has different meanings from that for the law of iterated logarithms. While the latter is an estimate for how far a typical path can go, the former is an estimate for averages, hence paths which are accidentally close to the origin at specified step must also be considered, and it seems (at present) that it cannot be proved without further assumptions. For the self-repelling walks, a geometric consideration similar to the reflection principle can be applied.

Theorem 7 ([5]) *For any $u > 0$, the self-repelling walk $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, has a displacement exponent $\nu = \nu_u$ given by (17);* $\displaystyle\lim_{k\to\infty} \frac{1}{\log k} \log \mathrm{E}[\ |W_k|^s\ ] = s\nu,\ s \geqq 0\,.$ $\diamond$

The known proof is techincally involved and we leave it to the original paper [5].

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