On the multi-string solutions of the self-dual static Einstein-Maxwell-Higgs system

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Abstract

In this report we present recent result on the existence and precise decay estimates at infinity of solutions to the Bogomol'nyi system of the static Einstein equations coupled with the Maxwell-Higgs fields with translational symmetry in one direction. The equations model cosmic strings(or superconducting strings) in equilibrium state. The Higgs fields of our solutions, in particular, tend to the symmetric vacuum at infinity. The construction of our solution is by the perturbation type of argument combined with the implicit function theorem.

1 Introduction

Let us consider the (3+1) dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, where $g_{\mu\nu}$ is a metric with signature given by $(-,+,+,+)$. We denote $g^{\mu\nu}$ for the inverse matrix of $g_{\mu\nu}$. We raise and lower the tensor indices by $g^{\mu\nu}$ and $g_{\mu\nu}$. On this manifold let us introduce the Lagragian,

$$\mathcal{L} = \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}g^{\mu\nu}(D_\mu\phi)(D_\nu\phi)^* + \frac{1}{8}(|\phi|^2-\sigma^2)^2, \tag{1.1}$$

where ϕ is a cross section on a $U(1)$-line bundle, called Higgs field, $A = A_\mu dx^\mu$ is a (gauge) connection 1-form, called the Maxwell field, $F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a (gauge) curvature 2-form, and

$D = d - iA$ is a (gauge) covariant derivative. We denote ϕ^* as the complex conjugation of ϕ. $\sigma > 0$ is called the symmetry breaking parameter. Let $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu})$ be the Christoffel symbol, representing the Levi-Civita connection on $(\mathcal{M}, g_{\mu\nu})$, and let

$$R^\mu_{\nu\rho\tau} = \partial_\tau\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\tau} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\tau\nu} - \Gamma^\mu_{\tau\alpha}\Gamma^\alpha_{\tau\nu}$$

be the Riemann curvature tensor on the manifold. Let $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ and $R = R^\alpha_\alpha$ be the Ricci tensor and the scalar curvature of the manifold respectively. Let $G > 0$ be the gravitational constant. Then, the Einstein equations coupled with the Maxwell-Higgs fields are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where the energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{2}[(D_\mu\phi)(D_\nu\phi)^* + (D_\nu\phi)(D_\mu\phi)^*] - g_{\mu\nu}\mathcal{L}, \tag{1.3}$$

coupled with the matter equations,

$$\frac{1}{\sqrt{|g|}}D_\mu(g^{\mu\nu}\sqrt{|g|}D_\nu\phi) = \frac{1}{2}(|\phi|^2 - \sigma^2)\phi, \tag{1.4}$$

and

$$\frac{1}{\sqrt{|g|}}\partial_\alpha(g^{\mu\nu}g^{\alpha\beta}\sqrt{|g|}F_{\nu\beta}) = \frac{i}{2}g^{\mu\nu}[\phi(D_\nu\phi)^* - \phi^*(D_\nu\phi)], \tag{1.5}$$

where we denoted $g = \det(g_{\mu\nu})$. We assume that our metric is static and translational invatiant along a spatial direction, say along the x_3 axis. More precisely, we assume our metric is of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx_3^2 + \gamma_{ij}dx^i dx^j,$$

where $\partial_t\gamma_{ij} = \partial_3\gamma_{ij} = 0$, and $\mathcal{M} = R^2 \times \mathcal{M}_2$. We also assume that our matter fields A_μ, ϕ depend on x_1, x_2, the coordinates of $\mathcal{M}_2$, and $A_\mu = (0, A_1, A_2, 0)$. We denote below $A = (A_1, A_2)$. In this case it is known([17],[22]) that the system (1.2)-(1.5) posess the self-dual equations,

$$K_\gamma = 8\pi G\mathcal{E}, \tag{1.6}$$

$$(D_j \pm i\varepsilon_j^k D_k)\phi = 0, \tag{1.7}$$

$$\varepsilon^{jk}F_{jk} \pm (|\phi|^2 - \sigma^2) = 0, \tag{1.8}$$

where K_γ is the Gaussian curvature of $(\mathcal{M}_2, \gamma_{ij})$, $\mathcal{E} = T_{00}$ is the energy density, ε_{jk} is the Levi-Civita skew-symmtetric tensor with the normalization $\varepsilon_{12} = \sqrt{\gamma}$, where $\gamma = \det(\gamma_{ij})$. The Bogomol'nyi system, (1.6)-(1.8) represents a model for cosmic strings(or superconducting strings) in equilibrium([11],[20]). We further assume that our reduced manifold, $(\mathcal{M}_2, \gamma_{ij})$ is conformally flat, namely there exists a function η such that

$$\gamma_{ij} = e^{\eta}\delta_{ij}. \tag{1.9}$$

Following [21], we make a scale transform, $x \mapsto \frac{x}{\sigma}, \phi \mapsto \sigma\phi, A_j \mapsto \sigma A_j$. Then, the energy and the Gaussian curvature transform as $\mathcal{E} \mapsto \sigma^4\mathcal{E}, K_\gamma \mapsto \sigma^2 K_\gamma$. Then, following standard Jaffe-Taubes' procedure[10], we represent

$$\phi = \exp\left(\frac{u}{2} + i\sum_{j=1}^{m} n_j Arg(z - z_j)\right),$$

where the zero set of ϕ, $\mathbb{Z}(\phi) = \{z_j\}_{j=1}^m \subset \mathbb{C} = \mathbb{R}^2$ is prescribed together with their multiplicities $\{n_j\}_{j=1}^m$. We can thus reduce further the system (1.6)-(1.9) into the semilinear elliptic system for (u, η)

$$\Delta u = e^{\eta}(e^u - 1) + 4\pi\sum_{j=1}^{m} n_j \delta(z - z_j), \tag{1.10}$$

$$\Delta(\eta + ae^u) = ae^{\eta}(e^u - 1), \tag{1.11}$$

where we set

$$a = 4\pi G\sigma^2. \tag{1.12}$$

The system (1.10)-(1.11) is our basic equations to solve in the following sections. We want to solve (1.10)-(1.11) under the finite energy condition

$$\int_{\mathbb{R}^2} \mathcal{E}e^{\eta}dx < \infty, \qquad \int_{\mathbb{R}^2} K_\gamma e^{\eta}dx < \infty. \tag{1.13}$$

Here we note that, in terms of u, η, $\mathcal{E}, K_\gamma$ and F_{12} have the representations,

$$K_\gamma = -\frac{1}{2}e^{-\eta}\Delta\eta = a\mathcal{E}, \qquad F_{12} = -\frac{1}{2}e^{\eta}(e^u - 1).$$

A solution pair (u, η) satisfying (1.10)-(1.11) generates a static finite energy solution (ϕ, A, g) of (1.6)-(1.8), (and thus solutions of (1.2)-(1.5)) called a multi-string solution. In particular, we consider the two types of solutions of (1.10)-(1.13) distinguished by the boundary conditions for u at infinity:

$$u(x) \to 0 \qquad \text{as} \qquad |x| \to \infty, \tag{1.14}$$

and

$$u(x) \to -\infty \qquad \text{as} \qquad |x| \to \infty. \tag{1.15}$$

Physically, (1.14) implies that the Higgs field, $\phi(x)$ has the asymmetric vacuum($|\phi(x)| = 1$) at infinity, while (1.15) implies that the Higgs field satisfies the symmetric vacuum($|\phi(x)| = 0$) at infinity. Mathematical study of the system (1.10)-(1.11) is extensively done in [7],[17],[22]. We also mention that recently there are many mathematical studies on the similar type of equations arising from other vortex models(See [2-6],[12-16],[18] and, in particular [21] for a comprehensive survey of the subject.). In [7] it is found that the necessary condition for existence of solution of (1.10)-(1.13) is $0 < aN < 2$, and under the assumption $0 < aN < 1$, general(nonradial) multi-string solutions satisfying (1.14) are constructed in [22]. In this paper, we construct a family of solution to (1.10)-(1.13) satisfying the condition (1.15) in the full range $0 < aN < 2$. Our method of construction is a variation of the perturbation type of argument, which has been developed in a series of papers [2-4]. In order to formulate our main theorem we introduce some functions. Given $\varepsilon > 0$, and $\delta \in \mathbb{C} = \mathbb{R}^2$, we define

$$\rho^{I}_{\varepsilon,\delta}(z) := \frac{8^{\frac{1}{a}} \varepsilon^{2N+2} \prod_{j=1}^{m} |z - z_j|^{2n_j}}{a^{\frac{1}{a}} (1 + |\varepsilon z + \delta|^2)^{\frac{2}{a}}}, \tag{1.16}$$

and

$$\rho^{II}_{\varepsilon,\delta}(z) := \frac{8\varepsilon^2}{a(1 + |\varepsilon z + \delta|^2)^2}. \tag{1.17}$$

where $z = x_1 + ix_2$. We also introduce the associated functions

$$\rho_1(r) := \frac{8^{\frac{1}{a}} r^{2N}}{a^{\frac{1}{a}} (1 + r^2)^{\frac{2}{a}}}, \tag{1.18}$$

and

$$\rho_2(r) := \frac{8}{a(1 + r^2)^2}, \tag{1.19}$$

where $r = |z|$. Below we set $f(t) = (a + 1)\rho_1(t)\rho_2(t)$. Then, the function $w_1(r)$ is defined by

$$w_1(r) := \varphi_0(r) \left\{ \int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1 - s)^2} ds + \frac{\phi_f(1) r}{1 - r} \right\} \tag{1.20}$$

with

$$\phi_f(r) := \left(\frac{1 + r^2}{1 - r^2} \right)^2 \frac{(1 - r)^2}{r} \int_0^r \varphi_0(t) t f(t) dt,$$

and

$$\varphi_0(r) := \frac{1 - r^2}{1 + r^2},$$

where $\phi_f(1)$ and $w_1(1)$ are defined as limits of $\phi_f(r)$ and $w_1(r)$ as $r \to 1$. We also define

$$w_2 := a w_1 - a \rho_1. \tag{1.21}$$

2 The Main Theorem

The following is our main result, the proof of which is in [1].

Theorem 2.1 *Let* $\{n_j\}_{j=1}^m \subset \mathbb{N}$ *and* $\{z_j\}_{j=1}^m \in \mathbb{R}^2$ *be given. We set* $N = \sum_{j=1}^m n_j$. *Suppose* $0 < aN < 2$. *Then, there exists a constant* $\varepsilon_1 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$ *there exists a family of solutions to (1.6)-(1.8),* $(\phi_\varepsilon, A^\varepsilon, \gamma_{ij}^\varepsilon)$ *satisfying the finite energy condition (1.13). Moreover, the solutions we constructed have the following properties:*

(i) The Higgs fields ϕ_ε *has zeros at* $\{z_j\}_{j=1}^m$ *with multiplicities* $\{n_j\}_{j=1}^m$ *respectively.*

(ii) The functions $\phi_\varepsilon, \gamma_{ij}^\varepsilon$ *have the representations*

$$\phi_\varepsilon(z) = \exp\left(\frac{u_\varepsilon}{2} + i\sum_{j=1}^m n_j Arg(z - z_j)\right), \tag{2.22}$$

and

$$\gamma_{ij}^\varepsilon = e^{\eta_\varepsilon}\delta_{ij}, \qquad i, j = 1, 2 \tag{2.23}$$

with

$$u_\varepsilon(z) = \ln \rho^I_{\varepsilon,\delta_\varepsilon^*}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 v_\varepsilon^*(\varepsilon z), \tag{2.24}$$

and

$$\eta_\varepsilon(z) = \ln \rho^{II}_{\varepsilon,\delta_\varepsilon^*}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 \xi_\varepsilon^*(\varepsilon z), \tag{2.25}$$

where $\delta_\varepsilon^* \to 0$ *as* $\varepsilon \to 0$, *and*

$$\begin{aligned} w_1(\varepsilon|z|) &= -\kappa_1 \ln|z| + O(1), & (2.26)\\ w_2(\varepsilon|z|) &= -\kappa_2 \ln|z| + O(1) & (2.27)\end{aligned}$$

as $|z| \to \infty$ *with*

$$\kappa_1 := \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{2+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}, \tag{2.28}$$

and

$$\kappa_2 := \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{1+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}. \tag{2.29}$$

The functions v_ε^* *and* ξ_ε^* *in (1.24), (1.25) satisfy*

$$\sup_{z\in\mathbb{R}^2} \frac{|v_\varepsilon^*(\varepsilon z)| + |\xi_\varepsilon^*(\varepsilon z)|}{\ln(|z|+1)} \le o(1) \qquad \text{as } \varepsilon \to 0. \tag{2.30}$$

(iii) There exist constants $C_1 = C_1(G,\sigma), C_2 = C_2(G,\sigma)$ *and functions* $\beta_1(\varepsilon)$, $\beta_2(\varepsilon)$ *defined on a small neighborhood of* $\varepsilon = 0$ *such that*

$$\ln|\phi_\varepsilon(z)|^2 = u_\varepsilon(z) = \left[2N - \frac{4}{a} - \beta_1(\varepsilon)\right]\ln|z| + o(\ln|z|) \quad \text{as } |z| \to \infty. \tag{2.31}$$

$$|D_1\phi_\varepsilon|^2 + |D_2\phi_\varepsilon|^2 \le \frac{C_1}{|z|^{\frac{4}{a}-2N+\beta_1(\varepsilon)}} + o\left(\frac{1}{|z|^{\frac{4}{a}-2N+\beta_1(\varepsilon)}}\right) \qquad \text{as } |z| \to \infty, \tag{2.32}$$

$$\eta_\varepsilon(z) = [-4 - \beta_2(\varepsilon)]\ln|z| + o(\ln|z|) \qquad \text{as } |z| \to \infty. \tag{2.33}$$

The Gaussian curvature has the decaying property,

$$\left|K_\gamma^\varepsilon(x) - \frac{a}{2}\right| = O(e^{u_\varepsilon - \eta_\varepsilon}) \qquad \text{as } |z| \to \infty, \tag{2.34}$$

and determined by comparison of decays between u_ε *and* η_ε *as described above. In the above the functions* $\beta_1(\varepsilon), \beta_2(\varepsilon)$ *satisfy*

$$\lim_{\varepsilon\to 0}\frac{\beta_1(\varepsilon)}{\varepsilon^2} = \kappa_1, \qquad \lim_{\varepsilon\to 0}\frac{\beta_2(\varepsilon)}{\varepsilon^2} = \kappa_2.$$

(iv) The corresponding magnetic flux, total gravitational curvature, and the energy of the matter part are given by

$$\int_{\mathbb{R}^2} F_{12}^\varepsilon dx = 4\pi\left(N - \frac{1}{a}\right) + \pi\kappa_1\varepsilon^2 + o(\varepsilon^2), \tag{2.35}$$

$$\int_{\mathbb{R}^2} K_\gamma^\varepsilon e^{\eta_\varepsilon} dx = 4\pi + \pi\kappa_2\varepsilon^2 + o(\varepsilon^2), \tag{2.36}$$

and

$$\int_{\mathbb{R}^2} \mathcal{E} e^{\eta_\varepsilon} dx = \frac{1}{G}\left[1 + \frac{\kappa_2}{4}\varepsilon^2 + o(\varepsilon^2)\right] \tag{2.37}$$

as $\varepsilon \to 0$ *respectively.*

Remarks:

(i) We note $\kappa_1, \kappa_2 > 0$ for $0 < aN < 1$, and $\kappa_1, \kappa_2 < 0$ for $1 < aN < 2$. Thus $aN = 1$ corresponds to the "critical" case similarly to the solutions constructed in [7],[22].

(ii) Even in the range $0 < aN < 1$ our multi-string solutions are different from those constructed in [22], since our solution satisfy the boundary condition (1.15), not (1.14).

(iii) We compare our decay estimates with the well-known results on the topological solutions in [19]. From (1.10) and (1.11) we find that

$$\Delta(au - \eta - ae^u - 2a\sum_{j=1}^{m} n_j \ln|z - z_j|) = 0.$$

Thus, for both the topological and the nontopological solutions we can set the harmonic function $h(z) = au - \eta - ae^u - 2a\sum_{j=1}^{m} n_j \ln|z - z_j| =$ Constant. Hence,

$$\lim_{|z|\to\infty} \frac{\eta(z)}{\ln|z|} = -2aN + a \lim_{|z|\to\infty} \frac{u(z)}{\ln|z|}. \tag{2.38}$$

The formula (1.38) holds for both the topological and the nontopological solutions. For the topological solutions, we have $\lim_{|z|\to\infty} \frac{u(z)}{\ln|z|} = 0$, and

$$\lim_{|z|\to\infty} \frac{\eta(z)}{\ln|z|} = -2aN,$$

which holds for general topological solutions. Namely, for any topological solution there should be obvious dependence of the decay of η on the total string number N. For the nontopological solutions, in particular, for our family of solutions $(u_\varepsilon, \eta_\varepsilon)$ constructed in Theorem 1.1, we derive from (1.31)

$$\lim_{|z|\to\infty} \frac{u_\varepsilon(z)}{\ln|z|} = 2N - \frac{4}{a} - \kappa_1\varepsilon^2 + o(\varepsilon^2),$$

Hence,

$$\lim_{|z|\to\infty} \frac{\eta_\varepsilon(z)}{\ln|z|} = -4 - a\kappa_1\varepsilon^2 + o(\varepsilon^2) = -4 - \kappa_2\varepsilon^2 + o(\varepsilon^2),$$

and, we obtain $\lim_{\varepsilon\to 0}\lim_{|z|\to\infty} \frac{\eta_\varepsilon(z)}{\ln|z|} = -4$, which has no dependence on N. This is not surprising, since, as will be clear in the next section, our solution η_ε is a perturbation of $\ln\rho_2$, which is smooth everywhere, and does not have any dependence on the vortices.

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