# §1 **Introduction.**

## • **Problem (P).**

$$
\text{(P)} \begin{cases} \partial u/\partial t = \Delta u + u^{p-1} & \text{in } \Omega \times (0, T_m), \\ u = 0 & \text{on } \partial\Omega \times (0, T_m), \\ u \geq 0 & \text{in } \Omega \times (0, T_m), \\ u(0) = a & \text{in } \Omega. \end{cases}
$$

where

· $u = u(x,t)$,

· $\Omega \subset \mathbb{R}^3$: ball,

· $\Delta = \sum_{j=1}^{3} \partial_{x_j}^2$: Laplacian,

· $p > 2$,

· $a$: continuous and
$a = 0$ on $\partial\Omega$,

· $T_m =$ the maximal existence
time of $u$.

It is known that

$$T_m < \infty \Rightarrow \lim_{t \to T_m} \|u(t)\|_\infty = \infty$$

where $\|u(t)\|_\infty = \max_{x \in \Omega} |u(x,t)|$: $L^\infty$-norm.

$$\partial u/\partial t = \Delta u + u^{p-1}$$

:" Reaction-Diffusion equation."

- **Linear part.**

$\partial u/\partial t = \Delta u$ (in $\Omega = \mathbb{R}^3$) $\Rightarrow$

$u(x,t) = \frac{1}{(4\pi t)^{3/2}} \int dy e^{-|y|^2/4t} a(y)$ for $t > 0$,

$\|u(t)\|_\infty \simeq \frac{1}{t^{3/2}}$ as $t \to \infty$.

- **Nonlinear part.**

$\partial u/\partial t = u^{p-1}$ (note that this is ODE) $\Rightarrow$

$u(t) = (\frac{\beta}{T-t})^\beta$, $\beta = 1/(p-2)$ for $t < T$,

$u(t) \simeq \frac{1}{(T-t)^\beta} \to \infty$ as $t \uparrow T$.

- **Structure of (P).**

| $\Delta u$ (diffusion term) | $u^{p-1}$ (blow up term) |
|---|---|
| makes $u$ flatten | makes $u$ peaking |

**Problem**

Behavior of solutions of (P) as $t \uparrow$?

$$\partial u/\partial t = \Delta u + u^{p-1}$$

- **Known facts.**

| $a$ | $u(t)$ | control equation |
|---|---|---|
| small | $u(t) \to 0$, $T_m = \infty$ | $u_t = \Delta u$ |
| large | $u(t) \to \infty$, $T_m < \infty$ | $u_t = u^{p-1}$ |

: well-known.

What happens when $a$: **middle**-size?

In general, it is **expected** that

$u(t) \to$ **stationary solution** as $t \to \infty$.

Really?

## §2 **Variational problem and stationary solutions.**

- **Stationary equation (E).**

$$\text{(E)} \begin{cases} 0 = \Delta u + u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

**Result**

$p < 6 \Rightarrow$ (E) has a (unique) solution.
$p = 6 \Rightarrow$ (E) has no solution.

What is 6?

- **Variational formulation for (E).**

$$J(u) = \frac{1}{2}\int |\nabla u(x)|^2 dx - \frac{1}{p}\int |u(x)|^p dx$$

· Derivative of $J$ at $u$ in the direction $v$:

$$\frac{d}{dt}J(u+tv)\Big|_{t=0} (= J'(u)v)$$
$$= \int (\Delta u(x) + u(x)^{p-1})v(x)dx.$$

Hence

$u$: critical point of $J$ ($J'(u) = 0$)

$\Updownarrow$

$u$: solution of (E).

• **Construction of the critical point of $J$.**

The construction of solutions of (E):

| | |
|---|---|
| **step 1.** | Construction of a sequence $(u_n)$ s.t. "nearly critical point (solution)". |
| **step 2.** | Convergence of $(u_n)$. |

When $p \leq 6$, **step 1.**: OK. For **step 2.**,

$$p < 6 \Rightarrow \text{OK}, \quad p = 6 \Rightarrow \text{NO}.$$

What happens for $(u_n)$ when $p = 6$?

• **Scale invariance of $J$.**

Let $\lambda > 0$ and $g_\lambda u(x) := \lambda^{1/2} u(\lambda x)$.

Then $\int |\nabla g_\lambda u|^2 = \int |\nabla u|^2$. Moreover,

$$\int |g_\lambda u|^p = \int |u|^p \Leftrightarrow p = 6.$$

$p = 6$ is the only number s.t.
$J$ is invariant under the scaling $(g_\lambda)$,
i.e., $J(u) = J(g_\lambda u), \quad \forall \lambda > 0, \ \forall u.$

**• Expression of the sequence consists of nearly critical point of $J$ when $p = 6$.**

· $U(x) = \frac{C}{(1+|x|^2)^{1/2}}$: sol. of

$$0 = \Delta u + u^{6-1} \text{ in } \underline{\mathbb{R}^3}. \quad (*)$$

· Rescale $U$ so that $g_\lambda U$ looks like

Then by the scale invariance,
$g_\lambda U$ satisfies (*) and nearly only defined on $\Omega$.

$\Rightarrow$ $g_\lambda U$ is a nearly c. point of $J$.

· Analyzing these facts further, we obtain

**Expression of the nearly c. point**

$(u_n)$ is a sequence of nearly c. point of $J$ $\Leftrightarrow$

$$u_n \simeq \sum_{j=1}^{m} g_{\lambda_n^j} U, \qquad \lambda_n^j \to \infty$$

for some natural number $m$.

i.e., some kinds of "quantization" occurs.

## • On “Quantization”.

· Quantum mechanical:
*discreteness* of the spectrum of operators which describes physical quantities.

· Here:
*scale invariance* and
*the uniqueness* of the solution of the limiting equation (*).

**Summary**

| | | |
|---|---|---|
| $p = 6$ | $\Rightarrow$ | $J$: scale-invariant |
| | $\Rightarrow$ | $(u_n)$: finite superposition of rescaled entire solutions |
| | $\Rightarrow$ | $(u_n)$ cannot converge. |

## §3 Asymptotic behavior of solutions of (P).

**• Problem.**

$\Omega =$ ball,

$a$: radial, nonincreasing,

$$\partial u/\partial t = \Delta u + u^{p-1}, \qquad u(t) \not\to 0 \text{ as } t \to \infty.$$

· When $p < 6$, $u(t) \to$ a unique solution of (E): reasonable.

· When $p = 6$, we have no solution of (E). Hence "$u(t) \to$ solution" cannot occur.

**True problem here.**

What happens for time-global solutions of (P) with $p = 6$?

**cf.** There are many many papers concerning the case $p < 6$. The analysis of the critical case, $p = 6$, is a long-standing open problem.

• **Known facts.**

Assume $p = 6$. Then

· $\|u(t)\|_\infty \to \infty$ as $t \to \infty$.

· For *some* $t_n \to \infty$ and $\lambda_n^j \to \infty$,

$$u(t_n, x) \simeq \sum_{j=1}^{m} (\lambda_n^j)^{1/2} U(\lambda_n^j x) \text{ in } H_0^1, \quad \lambda_n^j \to \infty.$$

Above results are far from satisfactory since

(1) It does not provide the detailed information about the behavior of norms $\|u\|_q := (\int_\Omega |u(x)|^q)^{1/q}$, i.e., the shape of the solution.

(2) It provides the information only for **some time sequence**.

Using the intersection comparison principle, we can verify that

**Theorem (M.I.)**

$p = 6 \Rightarrow$ there exists $\lambda(t) \to \infty$ such that

$$u(t) \simeq \lambda(t)^{1/2} U(\lambda(t)x) \text{ in } H_0^1,$$

moreover,

$$\|u(t)\|_q \to \begin{cases} \infty & \text{if} \quad q > 6, \\ \alpha & \text{if} \quad q = 6, \\ 0 & \text{if} \quad q < 6 \end{cases}$$

From this, we can see that

· The variational senerio actually controls such a infinite-time blow up.

· The role of the invariance norm is important. This norm plays a role of separatrix.

· The infinite-time blow up is totally different phenomena from the finite-time (i.e., ODE-like) blow up since in the latter blow up we have

$$\|u(t)\|_q \to \infty \text{ for any } q.$$