On Keel-Smith-Sogge Estimates and Quasilinear Wave Equations in Exterior Domains

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June 28, 2006

The Cauchy problem

- The Laplacian: $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$,
- d'Alembertian: $\Box = \partial_t^2 - \Delta$, $\Box_c = \partial_t^2 - c^2\Delta$
- The Cauchy problem

$$\begin{cases} \Box u = Q(u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases}$$

  - $u' = \partial_{t,x} u$,
  - Q is quadratic, linear in u''

- The multiple speed Cauchy problem

$$\begin{cases} \Box_{c_I} u^I = Q^I(u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad I = 1, 2, \dots, D \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases}$$

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Examples

- Einstein vacuum equations (quasilinear)
  - See Christodoulou-Klainerman ['93], Lindblad-Rodnianski ['05]
- Equations of Elasticity (quasilinear, multiple speed)
  - See John ['88], Klainerman-Sideris ['96], Sideris ['96, '00], Agemi ['00]
- Wave maps
- Minimal surface equation in the Minkowski metric
  - See Lindblad ['04], Brendle ['02]
- Born-Infeld equations
  - See Chae-Huh ['03]

Long-time existence

- Assume that f, g are "small" - "size" $\approx \varepsilon$.
- Long-time existence for the single speed case?

- Global existence for $n \geq 4$.
- Almost global existence, $n = 3$.
  - Lifespan $= \exp(c/\varepsilon)$.

Moreover, there are counterexamples (John ['81], Sideris ['83]) which show that blow up in finite time is possible for $n = 3$. To get global existence, we need to have some extra structure.

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Recipe of the proof

For now, let's restrict our attention to the single speed case with no assumption on the nonlinearity. Moreover, we shall focus on the semilinear case, though fairly standard modifications yield the quasilinear case.

Recipe:

- Invariant vector fields
- Energy estimates
- Pointwise decay estimates

Invariant vector fields

- (Space-time) Translations: $\partial_t, \partial_j, \quad j = 1, 2, \dots, n$
- Spatial Rotations: $\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \le i < j \le n$
- Scaling: $L = t\partial_t + r\partial_r$
- Lorentz Rotations: $\Omega_{0k} = x_k\partial_t + t\partial_k, \quad 1 \le k \le n$

Key observation:

$$[\Box, \partial] = [\Box, \Omega] = 0; \quad [\Box, L] = 2\Box.$$

Thus,

$$\Box u = 0 \implies \Box(\Gamma u) = 0$$

for any $\Gamma = \{\partial, \Omega, L\}$.

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Energy estimates

$$\|u'(t,\,\cdot\,)\|_2 \le \|u'(0,\,\cdot\,)\|_2 + \int_0^t \|\Box u(s,\,\cdot\,)\|_2\,ds$$

- Very standard
- Proof:
  - Integrate $\Box u \partial_t u$ over space-time.
  - Use integration by parts and the Cauchy-Schwarz inequality
- The proof is easily modified to allow for variable coefficient wave operators – which permits the study of quasilinear equations.
- Using the commutator estimates from above, we trivially obtain, for any multi-index α,

$$\|(\Gamma^\alpha u)'(t,\,\cdot\,)\|_2 \le \|(\Gamma^\alpha u)'(0,\,\cdot\,)\|_2 + \sum_{|\beta|\le|\alpha|} \int_0^t \|\Gamma^\beta \Box u(s,\,\cdot\,)\|_2 ds.$$

Decay estimates

Klainerman-Sobolev estimates: (Klainerman ['85])

$$(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|u(t,x)| \lesssim \sum_{|\alpha|\le\frac{n+2}{2}} \|\Gamma^\alpha u(t,\,\cdot\,)\|_2.$$

- Compare to Sobolev embedding: $H^s \subset L^\infty$, $s > \frac{n}{2}$. Or, ignoring fractional derivatives,

$$\|h\|_\infty \lesssim \sum_{|\alpha|\le\frac{n+2}{2}} \|\partial^\alpha h\|_2.$$

- Using Γ and a finer analysis yields decay.

Decay estimates

- In the proof, the $O(1/t^{\frac{n-1}{2}})$ decay is used to absorb the t-integral in the energy inequality.
- When $n \geq 4$, this is integrable and yields global existence.
- When $n = 3$, a $\log(t)$ appears which corresponds precisely to the exponential in the definition of almost global existence.

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The exterior domain problem

Let $\mathcal{K} \subset \mathbb{R}^n$ be a bounded, star-shaped set with smooth boundary.

We look to solve:

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In the discussion, we will focus on the scalar-valued semilinear equations, though the techniques are known to work for the more general case above.

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Problems when there is a boundary

- The problem is overdetermined.
  Solution: We must require that the data satisfy certain compatibility conditions.

Example

$$\begin{cases} \Box u = 0, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f; \quad \partial_t u(0,\,\cdot\,) = g. \end{cases}$$

Notice

$$\Delta u|_{\partial\mathcal{K}} = \partial_t^2 u|_{\partial\mathcal{K}} = 0.$$

Thus, we must, e.g., require that

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- The vector fields don't preserve the boundary conditions
  - ∂_t does preserve the boundary condition.
  - ∂_j, $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ almost do as their coefficients on bounded on $\mathcal{K}$.
  - $L = t\partial_t + r\partial_r$ has bounded normal component, but the coefficients can become large in an arbitrarily small neighborhood of the boundary.
  - $\Omega_{0k} = x_k\partial_t + t\partial_k$ is seemingly inadmissible for such boundary value problems. These are also a problem for multiple speed problems.

As such, we

- Cannot use the Klainerman-Sobolev inequalities
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Main result

Theorem

For the initial-boundary value problem above,

- $n = 3$, *Almost global existence*
  - *Keel-Smith-Sogge ['02,'04], M.-Sogge ['06]*
- $n \geq 4$, *Global existence*
  - *M. ['04], M.-Sogge [preprint, '06]*

Other studies of the nonlinear exterior domain problem

- Global existence exterior to a nontrapping obstacle $n \geq 6$: Shibata-Tsutsumi ['86]
- Global existence exterior to a ball, $n \geq 4$: Hayashi ['95]
- Null condition global existence exterior to a star-shaped obstacle, $n = 3$: Keel-Smith-Sogge ['02], M.-Sogge [preprint]
- Almost global existence for more general obstacles, $n = 3$: M.-Sogge ['05]
- Null condition global existence for more general obstacles, $n = 3$: M.-Sogge ['05], M.-Nakamura-Sogge ['05,'05]
- Almost global existence for star-shaped obstacles, $n = 3$: Kubo [preprint]

Recipe for Proof

The proof of the theorem is rather standard once the correct estimates have been established. Thus, these shall be our focus.

Outline of Proof:

- Decay estimates (Decay in $|x|$!)
- Energy estimates
- KSS estimates

Set $\Omega = \{\Omega_{ij} : 1 \le i < j \le n\}$ and $Z = \{\partial, \Omega\}$.

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Decay estimate - Weighted Sobolev

For $R \geq 1$,

$$\|h\|_{L^\infty(R/2<|x|<R)} \lesssim R^{-(n-1)/2} \sum_{|\alpha|\leq \frac{n+2}{2}} \|Z^\alpha h\|_{L^2(R/4<|x|<2R)}.$$

- Due to Klainerman ['86]
- Provides decay in $|x|$ (which exterior to a star-shaped obstacle is sufficient)
- Proof:
  - Apply Sobolev lemma on $\mathbb{R} \times S^{n-1}$
  - Compare the volume element with that of $\mathbb{R}^n$. The factor r^{n-1} accounts exactly for the decay.

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Standard energy estimates

Since $\partial_t u = 0$, the same proof as before yields

$$\|u'(t,\,\cdot\,)\|_2 \lesssim \|u'(0,\,\cdot\,)\|_2 + \int_0^t \|\Box u(s,\,\cdot\,)\|_2\,ds.$$

Moreover, as $[\Box, \partial_t] = 0$, we get

$$\|\partial_t^j u'(t,\,\cdot\,)\|_2 \lesssim \|\partial_t^j u'(0,\,\cdot\,)\|_2 + \int_0^t \|\partial_t^j \Box u(s,\,\cdot\,)\|_2\,ds.$$

Energy estimates involving translations

Idea: Use elliptic regularity to turn ∂^α into $\partial_t^{|\alpha|}$ modulo harmless terms.
In L^2, we have

$$\begin{aligned}\partial_x^2 &\to \Delta + \text{lower order terms}\\ &\to \partial_t^2 - \Box + \text{lower order terms}.\end{aligned}$$

$$\begin{aligned}\|\partial^\alpha u'(t,\cdot)\|_2 \lesssim \sum_{|\beta|\le|\alpha|} \|\partial^\beta u'(0,\cdot)\|_2 + \sum_{j\le|\alpha|}\int_0^t \|\partial_t^j \Box u(s,\cdot)\|_2\,dt\\ + \sum_{|\beta|\le|\alpha|-1} \|\partial^\beta \Box u(t,\cdot)\|_2.\end{aligned}$$

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Energy estimates involving Z

- Do the usual integration by parts with u replaced by $Z^\alpha u$.
- A boundary term results. Using a trace theorem and the fact that the coefficients of Z are $O(1)$ on $\partial\mathcal{K}$, this boundary term can be controlled by

$$\sum_{|\beta|\le|\alpha|+2} \|\partial^\beta u'\|_{L^2_t L^2_x([0,t]\times\{|x|\le 1\})}.$$

- Such terms mesh well with the KSS estimates which will follow.

KSS estimates

- Weighted $L^2_t L^2_x$ estimate. First used to study nonlinear problems by Keel-Smith-Sogge ['02].
- Closely related to previous estimates of, e.g., Mochizuki ['84], Strauss ['75], Ruiz-Vega ['94]
- Also closely related to certain well-known local smoothing estimates for the Schrödinger equation.

$$(\log(2+T))^{-1/2}\|\langle x\rangle^{-1/2}u'\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^n)} \lesssim \|u'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box u(s,\,\cdot\,)\|_2\,ds$$

KSS estimates

- For semilinear equations exterior to nontrapping obstacles, this (combined with the weighted Sobolev estimate and energy estimates) is sufficient to show almost global existence in $n = 3$ (Keel-Smith-Sogge ['02]) or global existence in $n \geq 4$ (M. ['04]).
- For the exterior of star-shaped obstacles, a variable coefficient version was shown and used to prove the above results for quasilinear equations (M.-Sogge ['06]). This relied heavily on techniques of Rodnianski ['05].
- A different version of a variable coefficient estimate was shown by Alinhac [preprint]

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To get a sense of such an application of this estimate, it

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should be compared to the standard energy estimate

$$\|u'(t,\,\cdot\,)\|_2 \le \|u'(0,\,\cdot\,)\|_2 + \int_0^t \|\Box u(s,\,\cdot\,)\|_2\,ds.$$

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KSS estimates - Proofs

At least three different proofs...

- By scaling, reduce to showing the estimate over $\{|x| < 1\}$.
  - Then use Plancherel and analyze the space-time Fourier transform
  - Or, in odd dimensions, use sharp Huygens' principle.
- Or use the multiplier method with multipler $\frac{r}{r+R}\partial_r$ where R runs over the dyadic numbers $\leq T$.

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