Distance and Convexity

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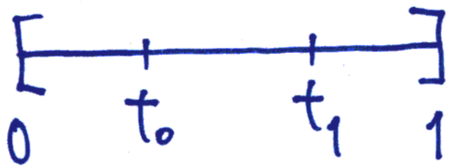
§ distance & geodesic

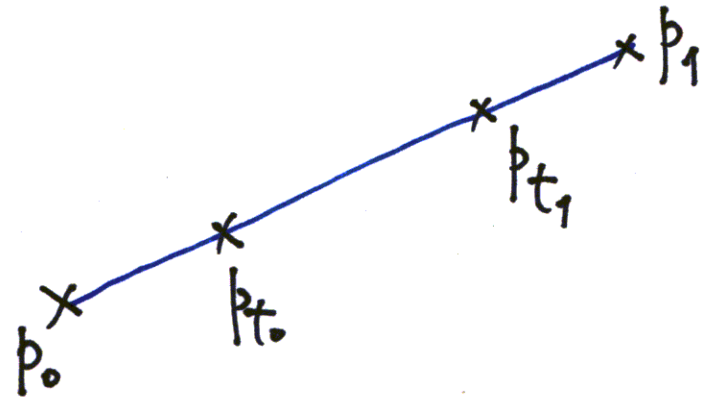
X : space, $\quad p_0, p_1$: points in X

$d(p_0, p_1)$: distance between p_0 & p_1

p_t, $0 \le t \le 1$: geodesic joining p_0 & p_1

i.e., $d(p_{t_0}, p_{t_1}) = d(p_0, p_1)\,|t_0 - t_1|$

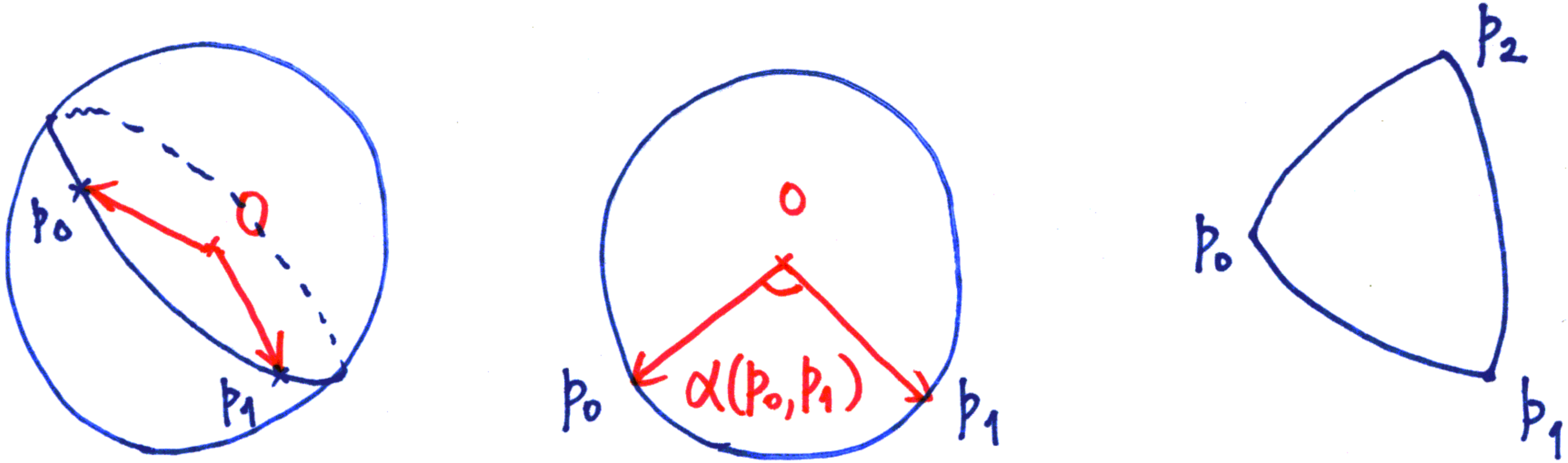




● X is called a geodesic space if any two points in X can be joined by a geodesic.

§ Sphere

$$S^n = \{ (x_1, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1 \} \ : \text{n-sphere (of radius 1)}$$



$d(p_0, p_1) = \alpha(p_0, p_1)$: distance between p_0 & p_1

geodesic = great circle.

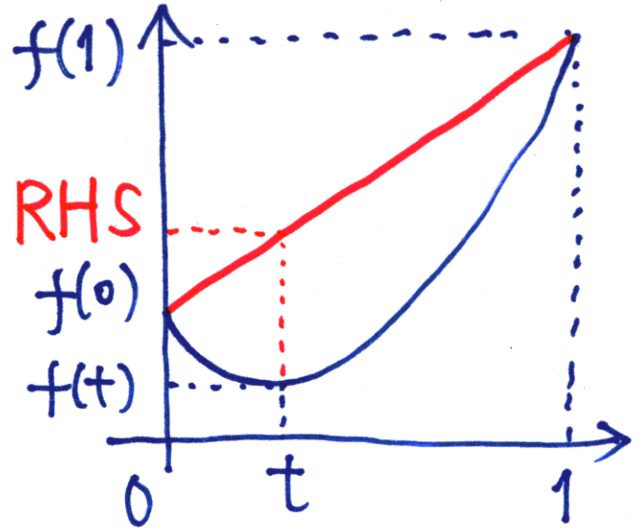
§ Convexity

A function $f(t)$, $0 \le t \le 1$

is convex

if $f(t) \le \underline{(1-t)f(0) + t f(1)}$

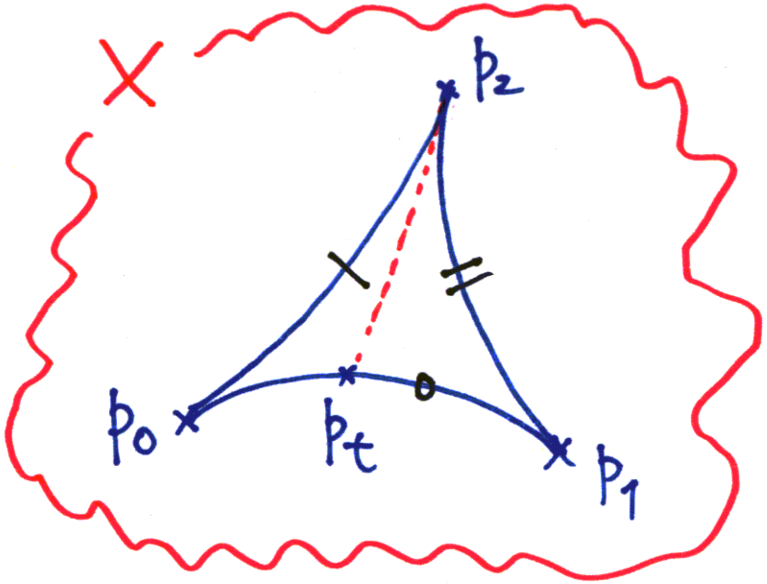
for all $0 \le t \le 1$.

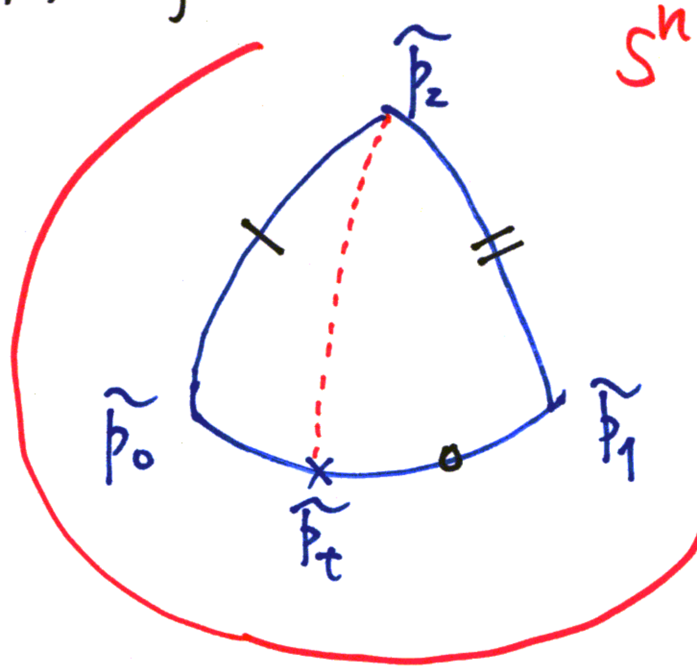


graph of $f(t)$ more curved than segments $f(0)f(1)$

§ CAT(1) property (Cartan, Alexandrov, Toponogov)

A geodesic space X is CAT(1) if





perimeter $< 2\pi$

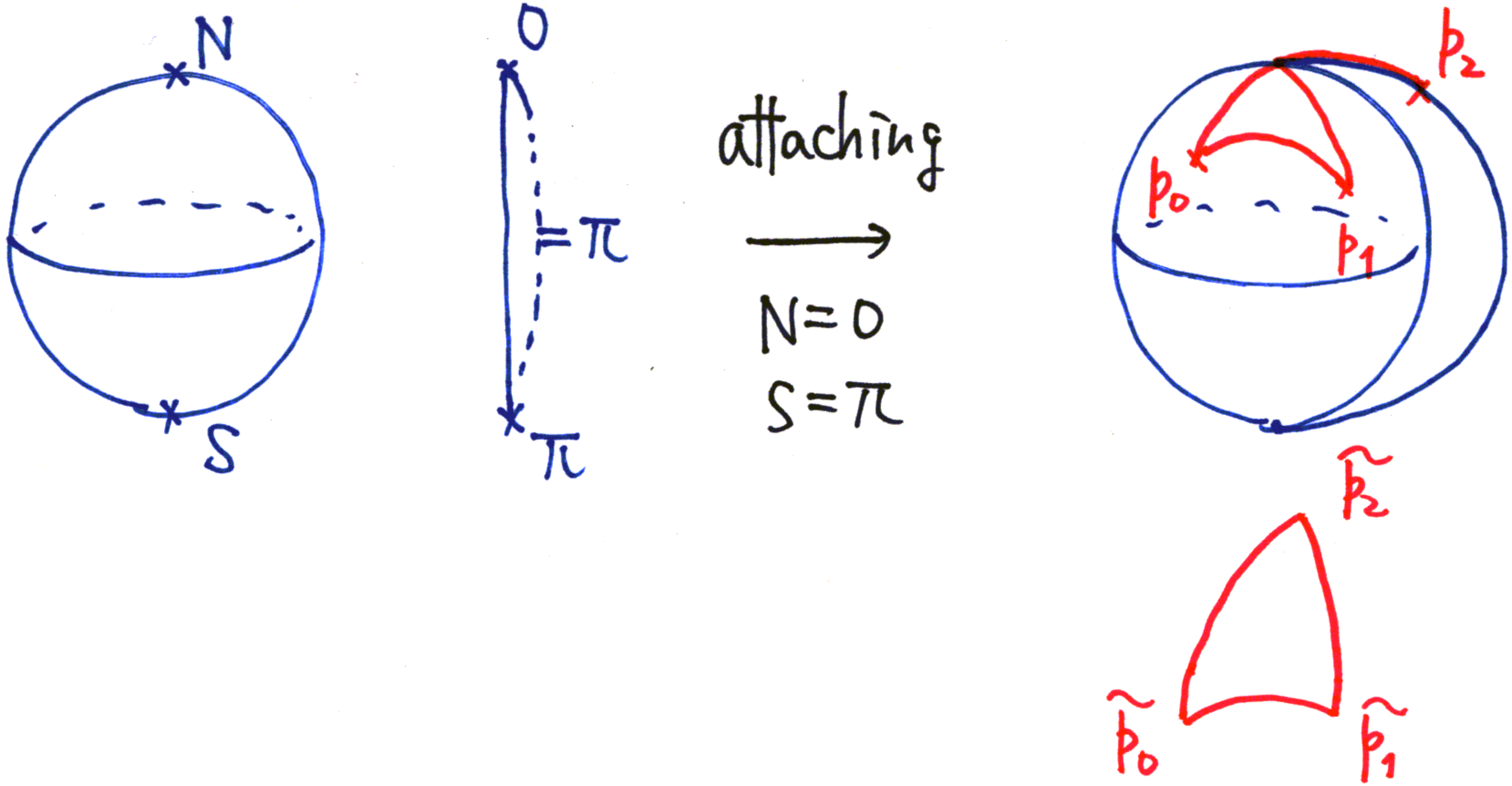
$$d(p_2, p_t) \le d(\tilde{p}_2, \tilde{p}_t) \quad \text{for all } 0 \le t \le 1$$

X more curved than S^n

§ Example

(0) S^n itself

(1) S^n

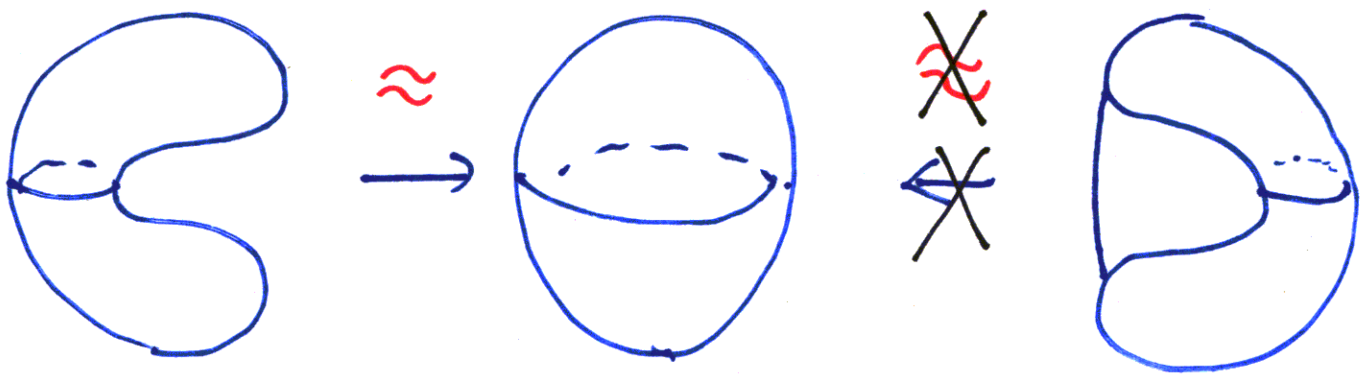


(2) Riemannian manifold
of sectional curvature ≤ 1
of injectivity radius $\geq \pi$.

§ homeomorphism

$A \approx B$ (homeomorphic)

continuously deformable from A to B.



<u>Poincaré conjecture</u> M : closed 3-manifold
simply connected

$$\overset{?}{\Longrightarrow} M \approx S^3.$$

§ A sphere theorem for CAT(1) spaces

with Alexander Lytchak (U. Bonn)

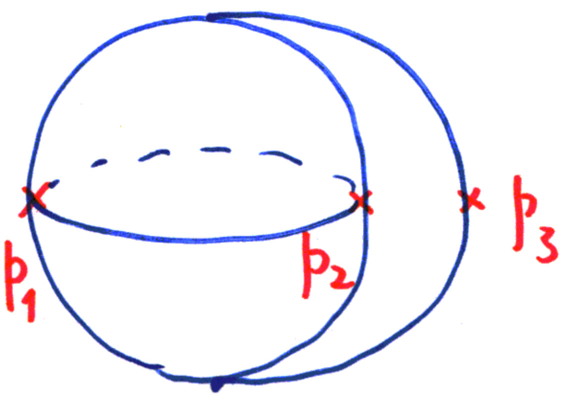
Theorem X: closed CAT(1) space

If no triple of points p_1, p_2, p_3 in X

with $d(p_i, p_j) \geq \pi$ for any $i \neq j$,

then $X \approx S^n$

$S^n \not\ni$



$d(p_i, p_j) \geq \pi$

$(=)$