The 21st Century COE Program of Tohoku University
Symposium
Exploring New Science by Bridging Particle-Matter Hierarchy

Mathematical Theory of Viscous Incompressible Fluid

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1. Navier-Stokes equations

$\mathbb{R}^3$: 3-D Euclidean space, $x = (x_1, x_2, x_3)$, $t \geq 0$: time

$$\begin{aligned} u &= u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \quad \text{velocity vector,} \\ p &= p(x,t) \quad \text{pressure} \end{aligned}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla = \frac{\partial}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} \quad \text{Lagrange differentiation}$$

(N-S)

$$\begin{cases} \dfrac{Du}{Dt} = \nu \Delta u - \dfrac{1}{\rho} \nabla p, \quad x \in \mathbb{R}^3, t > 0 \quad \text{(momentum conservation)} \\ \text{div } u = 0, \quad x \in \mathbb{R}^3, t > 0. \quad \text{(mass conservation)} \end{cases}$$

$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}, \quad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}), \quad \text{div } u = \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}$$

ν:kinematic viscosity, ρ:density, Assume that $\nu = \rho = 1$.

(1) $\quad u(x,0) = a(x) = (a_1(x), a_2(x), a_3(x))$ (initial data)

Cauchy Problem. For given a find a pair $\{u, p\}$ of functions satisfying (N-S) for $t > 0$ with (1) at $t = 0$.

(i) (**existence of global solutions**) For $a = a(x)$, dose (N-S) have a solution $\{u, p\}$ for all $t \geq 0$?

(ii) (**uniqueness & regularity of solutions**) Is the solution unique? Is the solution infinitely many times differentiable with respect to (x, t) ?

(iii) (**continuity of solutions for initial data**) Suppose that $\{v, q\}$ is another solution of (N-S) for the initial data $b(x)$. If $a \approx b$, then $\{u, p\} \approx \{v, q\}$?

If (i), (ii) and (iii) are affirmative, then we say that the Cauchy problem to (N-S) is **well-posed**.

(iv) (**blow-up of solutions**) Does there exist a finte time T_* such that the solution $\{u, p\}$ satifies

$$\begin{aligned} \{u(\cdot,t), p(\cdot,t)\} &\in C^\infty(\mathbb{R}^3) \quad \text{for } 0 < t < T_* \\ \{u(\cdot,t), p(\cdot,t)\} &\notin C^\infty(\mathbb{R}^3) \quad \text{for } T_* \leq t \text{ ?} \end{aligned}$$

(v) (**asymptotic behavior of solutions**) In case (i), $\lim_{t\to\infty} \{u(t), p(t)\} =$? In case (iii), $\lim_{t\to T_*} \{u(t), p(t)\} =$?

Millennium Prize Problem proposed by Clay Math. Inst. Are the questions (i) and (ii) true ?

Yes !

$\Longrightarrow$

$1,000,000（＝1億0432万円 3月1日22時49分現在）

cf. Poincaré Conjecture: another Millennium Prize Problem

東北大学「21世紀COE物質階層融合科学の構築」

春の学校

ポアンカレ予想，幾何化予想へのアプローチ

日時：3月22日〜25日　場所：東北大理学研究科川井ホール

世話人　小谷元子

Example of Cauchy problem to ODE

$$(2) \quad \begin{cases} \dfrac{dy(t)}{dt} = y(t), \quad t > 0, \\ y(0) = a \end{cases}$$

For $\forall a > 0$ $y(t) = e^t a$ is the solution for all $t > 0$ (**global solution**).

$$(3) \quad \begin{cases} \dfrac{dy(t)}{dt} = y(t)^2, \quad t > 0, \\ y(0) = a \end{cases}$$

$y(t) = \dfrac{a}{1 - at}$, $0 < t < 1/a$(**local solution**)

$\lim_{t \uparrow 1/a} y(t) = +\infty$(**blow-up at** $t = 1/a$)

(4)
$$\begin{cases} \dfrac{dy(t)}{dt} = y(t)^2 - y(t), \quad t > 0, \\ y(0) = a \end{cases}$$

解： $y(t) = \dfrac{1}{1 - (1 - 1/a)e^t}$

$0 < a < 1 \Longrightarrow y(t)$ is the **global solution** with $0 < y(t) < 1$ and $\lim_{t\to\infty} y(t) = 0$.

$a = 1 \Longrightarrow y(t) \equiv 1$ is the **trivial global solution**.

$1 < a \Longrightarrow y(t),\ 0 < t < \log\left(\dfrac{1}{1 - 1/a}\right)$ is the **local solution**,

$$\lim_{t\to\log\left(\frac{1}{1-1/a}\right)} y(t) = \infty \ (\textbf{blow-up})$$

Solutions to linear PDE

1. *Poisson equation*

$$-\Delta v = f, \quad x \in \mathbb{R}^3, \quad G(x) \equiv \frac{1}{4\pi}|x|^{-1}$$

$\Longrightarrow$

$$v(x) = \int_{\mathbb{R}^3} G(x-y)f(y)dy, \ \left(\int_{\mathbb{R}^3} \cdots dy \equiv \int\int\int_{\mathbb{R}^3} \cdots dy_1 dy_2 dy_3\right)$$

gives a solution formula.

2. *Cauchy problem to the heat equation*

$$\frac{\partial v}{\partial t} - \Delta v = f, \quad x \in \mathbb{R}^3, t > 0, \quad v(x,0) = b(x)$$

$\Longrightarrow$

$$v(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)b(y)dy + \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)f(y,\tau)dyd\tau,$$

gives a solution formula, where $\Gamma(x,t) \equiv (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$

Solution to nonlinear PDE $\Longrightarrow$ No solution formula!

Method 1; Linear perturbation

(N-S) $\approx$ perturbation from the linear Stokes equation

(N-S')
$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = -u \cdot \nabla u, \quad x \in \mathbb{R}^3, t > 0, \\ \text{div } u = 0 \quad x \in \mathbb{R}^3, t > 0, \\ u(x,0) = a(x) \end{cases}$$

$\Longleftrightarrow$ (Duhamel principle)

(IE)

$$u(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)a(y)dy - \int_0^t \int_{\mathbb{R}^3} E(x-y,t-\tau)u \cdot \nabla u(y,\tau)dyd\tau,$$

$$E_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} G(x-y)\Gamma(y,t)dy, \quad i,j = 1,2,3.$$

successive approximation(iteration method)

$$u^{(0)}(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)a(y)dy,$$

$$u^{(j+1)}(x,t) = u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}^3} E(x-y,t-\tau)u^{(j)} \cdot \nabla u^{(j)}(y,\tau)dyd\tau$$

$$(j = 1, 2, \cdots)$$

existence of solution $\iff$ $u(x,t) = \exists \lim_{j\to\infty} u^{(j)}(x,t)$

In general, only **local** solution can be constructed;

$\exists T_* < \infty$ such that $\exists \lim_{j\to\infty} u^{(j)}(x,t)$ for $0 \le t < T_*$

Method 2; Variational principle

Energy conservation

$$\frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|u_i(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(x,\tau)\right|^2 dxd\tau \tag{5}$$
$$= \frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|a_i(x)|^2dx$$

for all $0 \le t < \infty$. (5) is called **energy equality** of (N-S)-(1).

(5) $\Longrightarrow \exists$ weak solution u such that

$$\max_{0<t<\infty}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|u_i(x,t)|^2dx+\int_0^\infty\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(x,\tau)\right|^2 dxd\tau \le \int_{\mathbb{R}^3}\sum_{i=1}^{3}|a_i(x)|^2dx$$

advantage: $\exists u(\cdot,t)$ solution for all $0 < t < \infty$ (global solution)

disadvantage: smoothness of u is unknown!

Question: Can we control

$$(6)\quad \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^{3} |\Delta u_i(x,\tau)|^2 dx d\tau, \qquad \max_{t>0} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j}(x,t) \right|^2 dx$$

by means of the initial data a ?

2. Existence of global weak solution

$$L^2_\sigma \ = \ \{u = (u_1, u_2, u_3); \text{div } u = 0, \int_{\mathbb{R}^3} \sum_{i=1}^{3} |u_i(x)|^2 dx < \infty\},$$

$$H^1_\sigma \ = \ \{u = (u_1, u_2, u_3) \in L^2_\sigma; \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j}(x) \right|^2 dx < \infty\}$$

$$u, v \in L^2_\sigma \Longrightarrow (u, v) \equiv \int_{\mathbb{R}^3} \sum_{i=1}^{3} u_i(x) v_i(x) dx$$

$$u, v \in H^1_\sigma \Longrightarrow (u, v)_{H^1} \equiv (u, v) + (\nabla u, \nabla v), \quad \nabla u = \left(\frac{\partial u_i}{\partial x_j} \right)_{i,j=1,2,3}$$

L^2_σ, H^1_σ: Hilbert spaces $H^1_\sigma \subset L^2_\sigma$

PDE theory in functional analysis

solution $u(x,t) \Longleftrightarrow$ one parameter family of t with its value in L^2_σ and H^1_σ, i.e.,

X: Hilbert space(Banach space), u: $t \in [0,T) \mapsto u(\cdot,t) \in X$,

ODE$\Longrightarrow$ $X = \mathbb{R}^1, \mathbb{R}^3, \cdots$, **finite dimensional** vector space

PDE$\Longrightarrow$ $X = L^2, H^1, \cdots$, **infinite dimensional** function space

$\|\cdot\|_X$: the norm of X,

$$L^s(0,T;X) \equiv \{u : t \in (0,T) \mapsto u(t) \in X; \int_0^T \|u(t)\|_X^s dt < \infty\}, \quad 1 \le s < \infty$$

$$L^\infty(0,T;X) \equiv \{u : t \in (0,T) \mapsto u(t) \in X; \sup_{t\in(0,T)} \|u(t)\|_X < \infty\}$$

$$C^m([0,T);X)$$

$$\equiv \{u : t \in [0,T) \mapsto u(t) \in X, m\text{-times continuously differentiable};$$

$$\sup_{t\in[0,T)} \|\frac{d^m}{dt^m} u(t)\|_X < \infty\}$$

Definition 2.1. Let $a \in L^2_\sigma$. A function u is a **weak solution** of (N-S)–(1) on $(0,T)$ if

(i) $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$;

(ii) The identity

$$\begin{aligned} &\int_0^T \{-(u(t), \frac{\partial \Phi}{\partial t}(t)) + (\nabla u(t), \nabla \Phi(t)) + (u \cdot \nabla u(t), \Phi(t))\} dt \\ = \; &(a, \Phi(0)) \end{aligned}$$

holds for all $\Phi \in C^1([0,T];H^1_\sigma)$ with $\Phi(\cdot,T) = 0$. (u satisfies (N-S) in the sense of **distribution**.)

Remarks. (i) $\{u,p\}$ satisfies (N-S)–(1) in the usual sense(u: **classical solution**) $\Longrightarrow$ u is a weak solution.

Indeed, we have by integration by parts

$$
\begin{aligned}
\int_0^T (\frac{\partial u(t)}{\partial t}, \Phi(t))dt &= -\int_0^T (u(t), \frac{\partial \Phi}{\partial t}(t))dt + (u(T), \Phi(T)) - (u(0), \Phi(0)) \\
&= -\int_0^T (u(t), \frac{\partial \Phi}{\partial t}(t))dt - (a, \Phi(0)), \\
(-\Delta u(t), \Phi(t)) &= (\nabla u(t), \nabla \Phi(t)), \\
(\nabla p(t), \Phi(t)) &= -(p(t), \mathrm{div}\ \Phi(t)) = 0
\end{aligned}
$$

hold for all $\Phi \in C^1([0,T); H^1_\sigma)$ with $\Phi(T) = 0$.

(ii) Conversely, u is a weak solution of (N-S)–(1) on $(0,T)$ with the **second derivatives** on $\mathbb{R}^3 \times (0,T)$ $\Longrightarrow$ $\exists p(x,t)$ such that $\{u, p\}$ is a classical solution.

Theorem 2.1. (Leray) For arbitrary $a \in L^2_\sigma$ there exists a weak solution u of (N-S)–(1) on $(0,\infty)$ such that

(7) $\frac{1}{2}\|u(t)\|^2_{L^2} + \int_s^t \|\nabla u(\tau)\|^2_{L^2} d\tau \le \frac{1}{2}\|u(s)\|^2_{L^2}, \quad 0 \le s \le t < \infty$

(8) $\|u(t) - a\|_{L^2} \to 0, \quad \text{as } t \to +0,$

where $\|u\|_{L^2} = \sqrt{(u, u)}$.

We solved Problem (i) by introducing the notion of **weak solutions**.

Problem (ii) In the weak solution $u(x,t)$ in Theorem 2.1 unique ? Is $u(x,t)$ differentiable with respect to for (x,t) ?

partial answer: (7) guarantees smoothness of u to some extent.

Theorem 2.2. (Leray) Suppose that u is a weak solution of (N-S)–(1) on $(0,\infty)$ with the energy inequality (7). There is a disjoint family $\{I_k\}_{k=0}^{\infty}$ of intervals on $(0,\infty)$ such that

(i) $\exists T_0 > 0$ such that $I_0 = [T_0, \infty)$;

(ii) $|(0,\infty) \setminus \cup_{k=0}^{\infty} I_k| = 0$ and $\displaystyle\sum_{k=1}^{\infty} |I_k|^{\frac{1}{2}} < \infty$;

(iii) $u(\cdot, t) \in C^{\infty}(\mathbb{R}^3)$ for all $t \in I_k$, $(k = 0, 1, \cdots)$,

where $|I|$ denotes the length of the interval I.

Leray named the weak solution u of (N-S)-(1) with the energy inequality (7) a **turbulent solution**.

how to derive the energy inequality (7):

Suppose that $\{u, p\}$ is a classical solution. $\Longrightarrow$

$$(9) \quad (\frac{\partial u}{\partial t}, u) + (-\Delta u, u) + (u \cdot \nabla u, u) + (\nabla p, u) = 0, \qquad t > 0.$$

By integration by parts, we have

$$(\frac{\partial u}{\partial t}, u) = \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2, \qquad (-\Delta u, u) = \|\nabla u\|_{L^2}^2,$$

$$\begin{aligned}
(u \cdot \nabla u, u) &= \int_{\mathbb{R}^3} \sum_{i,j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} u_i dx \\
&= -\int_{\mathbb{R}^3} \sum_{i,j=1}^{n} \frac{\partial u_j}{\partial x_j} (u_i)^2 dx - \int_{\mathbb{R}^3} \sum_{i,j=1}^{n} u_j u_i \frac{\partial u_i}{\partial x_j} dx \\
&= -(\mathrm{div}\ u, |u|^2) - (u, u \cdot \nabla u) \\
&= -(u, u \cdot \nabla u) = 0, \\
(\nabla p, u) &= -(p, \mathrm{div}\ u) = 0.
\end{aligned}$$

Hence it follows from (9) that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0 \quad \text{for all } t > 0.$$

Integrating the above identity in t over the interval (s,t), we have

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \frac{1}{2}\|u(s)\|_{L^2}^2, \quad 0 \le s \le t < \infty$$

$\Longrightarrow$

Apriori estimate !

Uniqueness and regularity of weak solutions

Theorem 2.3. (von Wahl, Giga, Sohr–K.) Let $a \in L^2_\sigma$.

(i) (*uniqueness*) Let u and v be two weak solutions of (N-S)–(1) on $(0,T)$. Suppose that v satisfies the energy inequality (7) with $s = 0$. Assume that u satisfies

(10) $\quad u \in L^\infty(0,T;L^3), \quad$ i.e., $\quad \sup_{0<t<T} \int_{\mathbb{R}^3} |u(x,t)|^3 dx < \infty.$

Then we have $u = v$ on $\mathbb{R}^3 \times (0,T)$.

(ii) (*regularity*) Suppose that u is a weak solution of (N-S)–(1) on $(0,T)$. If

(11) $$u \in C([0,T);L^3),$$

i.e.,

$$t \in [0,T) \mapsto \|u(t)\|_{L^3} \equiv \left(\int_{\mathbb{R}^3} |u(x,t)|^3 dx\right)^{\frac{1}{3}} \in \mathbb{R} \quad \text{continuous function on } [0,T),$$

Then we have

$$\frac{\partial u}{\partial t}, \nabla u, \nabla^2 u, \cdots, \nabla^k u, \cdots \in C(\mathbb{R}^3 \times (0,T)).$$

Question. In the weak solution $u \in L^\infty(0,T;L^3)$ of (N-S)–(1) a smooth function ?

Recent Result. Iskauriaza-Seregin-Šverák showed

$$u \in L^\infty(0,T;L^3) \quad \Longrightarrow \quad u(t) \in C^\infty(\mathbb{R}^3), 0 < \forall t < T$$

by contradiction argument.

Problem. Direct proof of regularity result on weak solution in the class $L^\infty(0,T;L^3)$

Scaling invariance: $\lambda > 0$:parameter, a family $\{u_\lambda, p_\lambda\}$ of functions

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

$\{u, p\}$ is a solution of (N-S) on $\mathbb{R}^3 \times (0, \infty)$.

$\Longleftrightarrow$

$\{u_\lambda, p_\lambda\}_{\lambda>0}$ is a solution of (N-S) on $\mathbb{R}^3 \times (0, \infty)$.

It is easy to check that

$$\begin{aligned} \|u_\lambda\|_{L^\infty(0,\infty;L^3)} &= \sup_{0<t<\infty} \left(\int_{\mathbb{R}^3} |u_\lambda(x,t)|^3 dx)\right)^{\frac{1}{3}} dt \\ &= \sup_{0<t<\infty} \int_{\mathbb{R}^3} \left(|u(x,t)|^3 dx\right)^{\frac{1}{3}} dt \\ &= \|u\|_{L^\infty(0,\infty;L^3)} \end{aligned}$$

holds for all $\lambda > 0$. This implies that the space $L^\infty(0, \infty; L^3)$ is *invariant* under the change of scale such as $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$.

Importance!(*Fujita-Kato principle*) Find a solution u in a function space Y on $\mathbb{R}^3 \times (0, \infty)$ such as $\|u_\lambda\|_Y = \|us\|_Y$ holds for all $\lambda > 0$.

3. Local existence of classical solution.

Under which initial data a can we construct the weak solution u of (N-S)-(1) with (10) or (11) ?

$$
\begin{aligned}
L^r &\equiv \{u = (u_1, u_2, u_3); \|u\|_{L^r} = \left(\int_{\mathbb{R}^3} |u(x)|^r dx\right)^{\frac{1}{r}} < \infty\} \\
L^r_\sigma &\equiv \{u \in L^r; \text{div } u = 0\}
\end{aligned}
$$

Theorem 3.1. (Kato, Giga) Let $3 \le r < \infty$ and let $a \in L^r_\sigma$. Then there exist $T_* > 0$ and a unique solution u of (N-S)–(1) on $(0, T_*)$ such that

$$u \in C([0, T_*); L^r_\sigma) \tag{12}$$

$$\frac{\partial u}{\partial t}, \Delta u \in C((0, T_*); L^r_\sigma) \tag{13}$$

If in addition $a \in L^r_\sigma \cap L^2_\sigma$, then u is also a weak solution of (N-S)–(0.1) on $(0, T_*)$ with the energy equality (5) for $0 \le t \le T_*$.

Remark. (i) By (12) we see that $u(t)$ is a *classical* solution on $\mathbb{R}^3 \times (0, T_*)$.

(ii) T_*: time interval of local classical solution

(14) $$T_* = \frac{C}{\|a\|_{L^r}^{\frac{2r}{r-3}}} \quad \text{for } 3 < r < \infty,$$

where $C = C(r)$ is a constant independent of a.

$$\|a\|_{L^r} \ll 1 \Longrightarrow T_* \gg 1,$$
$$1 \ll \|a\|_{L^r} \Longrightarrow T_* \ll 1$$

(iii) Question: Can we represent T_* for $a \in L^3_\sigma$?

Corollary 3.2.(**global classical solution of small data**) There is $\delta > 0$ such that if $a \in L^3_\sigma$ satisfies $\|a\|_{L^3} \leq \delta$, then we have in Theorem 3.1 that $T_* = \infty$.

Question.

(i)(*continuation*) $u(t) \in C^\infty(\mathbb{R}^3)$ for $t \geq T_*$?

or

(ii) (*blow-up*) $\lim_{t \uparrow T_*} \|u(t)\|_{L^r} = \infty$?

Consider the vorticity rot $u \equiv \omega = (\omega_1, \omega_2, \omega_3)$, where

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

Theorem 3.3. (Ogawa-Taniuchi-K.,Yatsu-K.) Let $a \in L^r_\sigma$, $3 \le r < \infty$. Suppose that u is a solution of (N-S)–(1) on $(0, T_*)$ with (12) and (13). If

$$\int_0^{T_*} \|\omega_i(t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \quad i = 1, 2, 3, \tag{15}$$

or

$$\int_0^{T_*} \|\omega_i(t)\|_{BMO} dt < \infty, \quad i = 1, 2, \tag{16}$$

then there exists $T' > T_*$ such that u can be extended to the solution on $(0, T')$ of (N-S)–(1) as

$$u,\ \frac{\partial u}{\partial t},\ \Delta u \in C(0, T'); L^r_\sigma). \tag{17}$$

Remarks. (i) Beale-Kato-Majda showed that if

(18) $$\int_0^{T_*} \|\omega_i(t)\|_{L^\infty} dt < \infty, \quad i = 1, 2, 3,$$

then $\exists T' > T_*$ such that (18) holds. Notice that

$$\|\omega\|_{\dot{B}^0_{\infty,\infty}} \le C\|\omega\|_{BMO} \le C\|\omega\|_{L^\infty}, \quad \|\omega\|_{L^\infty} \equiv \sup_{x \in \mathbb{R}^3} |\omega(x)|.$$

(ii) Vortex equation in $\mathbb{R}^3$

$$\frac{\partial \omega}{\partial t} - \Delta\omega + u \cdot \nabla\omega - \omega \cdot \nabla u = 0$$

On the other hand, in $\mathbb{R}^2$ for $u = (u_1, u_2)$ we have

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} : \quad \text{scalar function}$$

with

$$\frac{\partial \omega}{\partial t} - \Delta\omega + u \cdot \nabla\omega = 0.$$

Maximum principle $\Longrightarrow$

$$\sup_{0<t<T} \|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \le \|\text{rot } a\|_{L^\infty(\mathbb{R}^2)}. \quad \text{(18) is always OK.}$$

(iii) The criterion (15) holds also for the equation of perfect fluids, i.e., the Euler equations.

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \quad x \in \mathbb{R}^3, t > 0 \\ \text{div } u = 0, \quad x \in \mathbb{R}^3, t > 0. \end{cases}$$

Question. Does the criterion (16) hold also for (E) ?