対称性の視点による
ソリトン方程式とパンルヴェ方程式

Soliton equations and Painlevé equations
through symmetry

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Soliton equations	$\longrightarrow$	Painlevé equations
(nonlinear PDE)	Reduction	(nonlinear ODE)
$\Uparrow$		$\Uparrow$
Affine Lie algebra		Affine Weyl group

1. Generalization of the construction of soliton eqs.
2. Lie algebraic description of reduction.

Affine Lie algebra:

$$\hat{\mathfrak{g}} = \left\{ X(z) = \sum_{i=-\infty}^{\infty} X_i z^i \;\middle|\; X_i : \text{matrix} \right\}$$

$\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2 \;\leftrightarrow\; X_i : 2 \times 2$ matrix, $\mathrm{tr} X_i = 0$.

$\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_3 \;\leftrightarrow\; X_i : 3 \times 3$ matrix, $\mathrm{tr} X_i = 0$.

Soliton equations $\rightarrow$ Painlevé equations.

Ex) modified KdV $\rightarrow$ Painlevé II

modified KdV : $q_t - \frac{1}{4} q_{xxx} + \frac{3}{2} q^2 q_x = 0$

$\downarrow$ Similarity reduction $\lambda q(\lambda x, \lambda^3 t) = q(x,t)$

Painlevé II : $q'' = 2q^3 + xq + \alpha$ (α: integral constant)

• mKdV $\leftrightarrow$ spectrum preserving deformation

$$\frac{\partial Y}{\partial x} = B_1(z)Y, \; B_1(z) = \begin{bmatrix} q & 1 \\ z & -q \end{bmatrix}$$

$$\frac{\partial Y}{\partial t} = B_3(z)Y, \; B_3(z) = z\begin{bmatrix} q & 1 \\ z & -q \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -q' - q^2 \\ z(q' - q^2) & 0 \end{bmatrix}$$

compatibility $\Leftrightarrow \dfrac{\partial B_1(z)}{\partial t} = \dfrac{\partial B_3(z)}{\partial x} - [B_1(z), B_3(z)]$

(Lax equation)

$\downarrow$ Similarity reduction $\lambda q(\lambda x, \lambda^3 t) = q(x,t)$

• $\mathrm{P_{II}}$ $\leftrightarrow$ monodromy preserving deformation equation.

$$z\frac{\partial Y}{\partial z} = A(z)Y, \; \frac{\partial Y}{\partial x} = B_1(z)Y, \; A(z) = \frac{\sigma^z}{4} + \frac{xB_1}{2} - 2B_3,$$

compatibility $\Leftrightarrow \dfrac{\partial A(z)}{\partial x} = z\dfrac{\partial B(z)}{\partial z} - [A(z), B(z)]$

Monodromy

Ex) $\quad z\dfrac{dy}{dz} = \alpha y \;\Leftrightarrow\; y = z^{\alpha}$ ($z = 0$: singularity)

$z = re^{i\theta}$, $r > 0$

θ	0	$\longrightarrow$	2π
y	z^{α}	$\longrightarrow$	$e^{2\pi i\alpha} z^{\alpha}$

Ex) $\quad z\dfrac{dY}{dz} = A(z)Y \;\Leftrightarrow\; Y = Y(z)$

($z = 0, \ldots$: singularity)

z	z_0	$\longrightarrow$	$z_0 e^{2\pi i}$
Y	$Y(z_0)$	$\longrightarrow$	$Y(z_0)\Gamma_0$

, $\quad \ldots$

Continuous deformation of parameter x
preserving the monodromy matrix

$$\Leftrightarrow \frac{\partial A(z)}{\partial x} = z\frac{\partial B(z)}{\partial z} - [A(z), B(z)]$$

1 Affine Lie algebraic approach to the soliton equations

Lax equations of the soliton equations

$$\frac{\partial B_m(z)}{\partial t_n} = \frac{\partial B_n(z)}{\partial t_m} - [B_m(z), B_n(z)], \quad B_m(z), B_n(z) \in \hat{\mathfrak{g}}$$

→ Drinfel'd-Sokolov hierarchy

[Drinfel'd, Sokolov] (1985),

→ generalized Drinfel'd-Sokolov hierarchy

[de Groot, Hollowood, Miramontes] (1992),

→ Generalization of the generalized DS hierarchy

[Kakei, K] (2004)

$$\left\{\begin{array}{l} \bullet \text{ affine Lie algebra } \hat{\mathfrak{g}} \\ \bullet \text{ Heisenberg subalgebra} \\ \bullet \text{ Gauss decomposition} \end{array}\right\} \Rightarrow \quad \text{soliton equations}$$

Ex) $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, (p) = principal, (h) = homogeneous.

Gauss decomp.

Heisenberg subalg.

	(p)	(h)
(p)	mKdV	KdV
(h)	∂NLS	NLS

KdV : Korteweg-de Vries eq. $u_t + u_{xxx} + uu_x = 0$

mKdV : modified KdV eq. $q_t + q_{xxx} + q^2 q_x = 0$

NLS : nonlinear Schrödinger eq. $\mathrm{i}q_t = \frac{1}{2}q_{xx} + |q|^2 q$

∂NLS : derivative NLS $\mathrm{i}q_t = \frac{1}{2}q_{xx} + 2\mathrm{i}q^2\bar{q}_x + 4|q|^4 q$

2 Lie algebraic description of similarity reduction

	Heis.	soliton eq.	reduction
$\widehat{\mathfrak{sl}}_2$	principal	mKdV	$\rightarrow$ P_{II}
	homogeneous	NLS	$\rightarrow$ P_{IV} (1)
$\widehat{\mathfrak{sl}}_3$	principal	mBoussinesq	$\rightarrow$ P_{IV}
	$2+1$	mYajima-Oikawa	$\rightarrow$ P_{V} (2)
	homogeneous	3wave, cNLS	$\rightarrow$ P_{VI} (3)

(1) (with Kakei) Int. Math. Res. Not. **2004**, no. 78, 4181–4209 (2004)

(2) (with Ikeda, Kakei) J. Phys A: Math. Gen. **36**, no. 45, 11465–11480 (2003)

(3) (with Kakei) (in preparation)

Summary

- Generalization of the generalized Drinfel'd-Sokolov hierarchy.
- Lie algebraic description of similarity reduction

Soliton equations	$\longrightarrow$	Painlevé equations
(nonlinear PDE)	Reduction	(nonlinear ODE)
$\Uparrow$		$\Uparrow$
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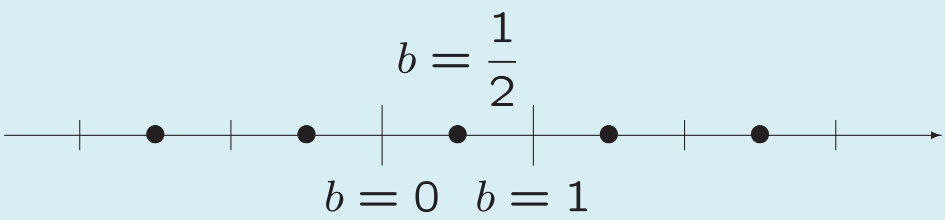
cf) **Affine Weyl group symmetry of the Painlevé equations**

Ex) $\mathrm{P_{II}}(b)$: $y'' = 2y^3 + xy + b - \dfrac{1}{2}$

y : a solution of $\mathrm{P_{II}}(b)$

$\Rightarrow S(y) = y + \dfrac{b}{y' + y^2 + x/2}$ is a solution of $\mathrm{P_{II}}(-b)$,

$T(y) = -y + \dfrac{b-1}{y' - y^2 - x/2}$ is a solution of $\mathrm{P_{II}}(1-b)$.



[Okamoto](1986) $\mathrm{P_{II}}, \mathrm{P_{III}}, \mathrm{P_{IV}}, \mathrm{P_{V}}, \mathrm{P_{VI}}$ have affine Weyl group symmetries cf)

- 2-point correlation function of the scaling limit of the 2D Ising model [Wu, McCoy, Tracy, Barouch] (1973 –1976)

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{2}\sinh 2u$$

$$t = r/2, y = e^{-u} \quad \Uparrow \quad \alpha = \beta = 0, \gamma = -\delta = 1$$

- Painlevé III equation

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$