# Single-point condensation phenomena for a four-dimensional biharmonic semilinear problem

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## Problem.

4-th order elliptic problem $(E_p)$:

$$(E_p)\begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

- $\Omega$ is a smooth bounded domain in $\mathbf{R}^4$,
- $\Delta^2 = \Delta\Delta$ is an iterated Laplacian (biharmonic operator) in $\mathbf{R}^4$,
- $p > 1$ is any positive number.

## Background

- Mathematical biology,
- Conformal geometry on 4-manifold.

## Question

What happens to solutions of $(E_p)$ when we take $p \to \infty$ ?

How do the normalized solutions look like when $p \to \infty$ ?

# Least energy solution

- Constrained minimization problem:

$$C_p^2 := \inf\{ \int_\Omega |\Delta u|^2 dx : \\ u \in H^2 \cap H_0^1(\Omega), \|u\|_{p+1} = 1 \}$$

$\exists\ \underline{u}_p \in H^2 \cap H_0^1(\Omega), \|\underline{u}_p\|_{p+1} = 1$ :

minimizer for $C_p^2$.

$$u_p := C_p^{\frac{2}{p-1}} \underline{u}_p.$$

$u_p$ : **least energy solution** to $(E_p)$.

## Main results.

### Theorem 1.

*Assume $\Omega$ is a smooth bounded convex domain in $\mathbf{R}^4$ and $u_p$ is the least energy solutions to $(E_p)$.*

*Then we have :*

$$1 \leq \liminf_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} \leq$$

$$\leq \limsup_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} \leq \sqrt{e}.$$

- Normalized function :

$$w_p(x) := \frac{u_p(x)}{\int_\Omega u_p^p dx}.$$

- $w_p$ satisfies

$$\begin{cases} \Delta^2 w_p = f_p(x) := \frac{u_p^p(x)}{\int_\Omega u_p^p dx} \quad \text{in } \Omega, \\ w_p > 0 \quad \text{in } \Omega, \\ w_p|_{\partial\Omega} = \Delta w_p|_{\partial\Omega} = 0. \end{cases}$$

**Definition (Blow up set of $\{w_{p_n}\}$)**

$$S := \{x \in \overline{\Omega} : \exists \text{a subsequence } w_{p_n'},$$
$$\exists \{x_n\} \subset \Omega \text{ such that}$$
$$x_n \to x \text{ and } w_{p_n'}(x_n) \to \infty\}.$$

**Definition (Peak set of $\{u_{p_n}\}$)**

$$P := \{x \in \overline{\Omega} : u_{p_n}(x) = \|u_{p_n}\|_{L^\infty(\Omega)}\}.$$

Fact

$\{\text{Peak points of } u_{p_n}\} \subset \{\text{Blow up points of } w_{p_n}\}$

## Theorem 2.

*Let* $\Omega \subset \mathbf{R}^4$ *be a smooth convex bounded domain.*

*Then for any sequence* $w_{p_n}$ $(p_n \to \infty)$,

$\exists$ *a subsequence such that :*

- *the blow up set* $S$ *of this subsequence satisfies* $S = \{x_0\}$ *for* $x_0 \in \Omega$

**(one point blow up)**, *and*

**(1)** (Convergence of $f_n$)

$$f_n(x) := \frac{u_{p_n}^{p_n}(x)}{\int_\Omega u_{p_n}^{p_n} dx} \xrightarrow{*} \delta_{x_0}$$

*in the sense of Radon measures of* $\Omega$.

**(2)** (Convergence of $w_{p_n}$)

$$w_{p_n} \to G_4(\cdot, x_0) \quad in\, C^4_{loc}(\overline{\Omega} \setminus \{x_0\})$$

*where*

$G_4(x, y)$ *denotes the Green function of* $\Delta^2$ *under the Navier boundary condition:*

$$\begin{cases} \Delta_x^2 G_4(x, y) = \delta_y(x),\ x \in \Omega, \\ G_4(x, y)|_{x \in \partial\Omega} = \Delta G_4(x, y)|_{x \in \partial\Omega} = 0. \end{cases}$$

**(3)** (Characterization of blow-up point)

$x_0$ *is a critical point of the Robin function* $R_4(x) = H_4(x,x)$ :

$$\nabla R_4(x_0) = \vec{0}.$$

*where*

$$H_4(x,y) := G_4(x,y) + \frac{1}{8\pi^2}\log|x-y|$$

*denotes the regular part of* $G_4$.

## Conclusion.

The set of peak points of $\{u_{p_n}\}$ is the same as the blow up set of $\{w_{p_n}\}$ and the least energy solutions must develop single-point spiky pattern, under the assumption that the domain is convex.

## Remark.

- The convexity of $\Omega$ is needed for the use of Method of Moving Planes (MMP).
- Application of Kelvin transformations does not work for our $\Delta^2$ case.

## Open problems.

(1) Do least energy solutions have multiple peaks if $\Omega$ is not convex ?

(2) Is the peak point $x_0$ of least energy solutions actually the maximum point of the Robin function ?

(3) Are least energy solutions to $(E_p)$ unique on convex domains ?