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Exploring New Science by Bridging Particle-Matter Hierarchy

Short-term Foreign Researchers

Research Report

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Blow-up solution for nonlinear Schrodinger equation with harmonic potential

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Please write a research report of one or more pages and submit it with this cover to your host researcher till the end of this March.

Research Report on Short-term Stay of Foreign Researcher Associated with the 21st Century COE Project entitled Exploring New Science by Bridging Particle-Matter Hierarchy in Graduate School of Science, Tohoku University

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I discussed the nonlinear Schrödinger equation with harmonic potential with my host professor Tsutsumi and Dr. Fukuizumi during my stay in Tohoku University. This system is regarded as a model equation for the Bose-Einstein condensate. The L^2 mass concentration in the blowup phenomena describes the collapse of the Bose-Einstein condensate. The analysis of this model is important from both mathematical and physical points of view.

When I was back in my university, with a student of mine, I put together a paper based on our research in Sichuan Normal University and partly on the discussions in Tohoku University. I attach a copy of our paper to this research report.

My one-week stay in Tohoku University was very fruitful and I do hope that this project will be a great success.

L^2-MASS CONCENTRATION OF BLOW-UP SOLUTION FOR NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL *

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Abstract: For the nonlinear Schrödinger equation with harmonic potential which describes Bose-Einstein condensate, the L^2-mass concentration properties of the blow-up solutions are obtained. Moreover, for arbitrary k points in R^n, a blow-up solution such that an L^2-mass concentration phenomenon occurs at the corresponding other k points is constructed.

Keywords L^2-mass concentration, Nonlinear Schrödinger equation, Blow-up solution, Harmonic potential.

MR(1991) subject classification 35Q55,35A22,35B30.

1 Introduction

In this paper, we study the Cauchy problem of the following nonlinear Schrödinger equation with harmonic potential

$$iu_t = -\frac{1}{2}\Delta u + \frac{1}{2}\omega^2|x|^2 u + a|u|^{\frac{4}{n}}u, \quad t \geq 0, x \in R^n, \tag{1.1}$$

$$u(0,x) = \varphi(x), \tag{1.2}$$

where $\omega > 0, a < 0$, are the parameters, n is the space dimension,$u(t,x) : R \times R^n \to C, \Delta$ is the Laplace operator on R^n, $\varphi(x)$ is the initial data. Equation (1.1) is well known as a model for describing the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap [2,3,16,19]. The harmonic potential $|x|^2$ models a magnetic

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field whose role is to confine the movement of particles, ω is the trap frequency, $\frac{a}{4\pi}$ is the scattering length[2,3,16,19].

Oh[15] has established the local well-posedness of the Cauchy problem (1.1), (1.2)in the corresponding energy field. Cazenave[4] and Zhang[19] studied the blow-up properties of the solutions and obtained the sufficient condition for blow-up solutions. At the same time, since the blow-up properties of the solutions correspond to the wave collapse of the condensate, physicists are very interested in the related studies[2,16]. This stimulates us to carry out elaborated mathematical research on the blow-up properties of the solutions.

We recall the classic nonlinear Schrödinger equation

$$iu_t = -\frac{1}{2}\Delta u + a|u|^{\frac{4}{n}}u, \quad t \geq 0, x \in R^n, \tag{1.3}$$

$$u(0,x) = \varphi(x), \tag{1.4}$$

where $a < 0$ is a parameter. There has been a lot of work on the blow-up properties of equation (1.3)[1,3,4,7,10–14,17,18,20]. Glassey[7] and Weinstein[17] studied the sufficient condition for the blow up solution to equation (1.3). Based on Weinstein[18], Merle and Tsutsumi[14] proved that the blow-up solution of (1.3) with a spherically symmetric initial data has the property of L^2-mass concentration. Moreover, Merle generalized this result in [13], that is , for any k points in R^n , he constructed a blow-up solution such that an L^2-mass concentration phenomenon occurs just at the k points. We see that for equation (1.3), a series of elaborated results on the blow-up have been got, particularly the construction of the blow-up solutions with some specific properties has been highly valued[1,13,18].

For equation (1.1), based on the need of physics, we hope to carry out the similar study as Merle and Tsutsumi on equation (1.3). However, due to the influence from harmonic potential $|x|^2$, there are differences in essence between equation (1.1) and equation (1.3) in some aspects such as the existence and stability of standing waves with ground state[5,6,8,19]. Moreover, we find that the method of [13] can not be used to equation (1.1). Fortunately, as a bridge between the solution of equation (1.1) and the solution of equation (1.3), the transform provided by Carles[3] is helpful for us. But this idea is also invalid unless it is based on the fact that we turn the limit of integral on the neighborhoods of moving point into the limit of integral on the neighborhoods of a fixed point. This technical treatment of the limit constitute the crucial part of this paper.

In this paper, it is proved that if the solution of the Cauchy problem (1.3),(1.4) blows up in finite time, the solution of the Cauchy problem(1.1),(1.2)will not only blow up but has the property of L^2-mass concentration at blow-up time. Specially, if initial data has a spherical symmetry, the L^2 -mass concentration phenomenon will occur at origin 0. Furthermore, for any k points $x_1, x_2, .., x_k$ in R^n , a blow-up solution is constructed such that an L^2-mass concentration phenomenon occurs at the corresponding other k points $cx_1, cx_2, .., cx_k$, where c is a constant that depends on the given k points. For these particular solutions, we also study their local behavior at the points at blow-up time. We notice that these results, especially for construction of blow-up solution with k L^2-mass concentration points , are different from Merle[13]. Moreover, we see that the results above describe the wave collapse phenomenon of Bose-Einstein Condensate from mathematical viewpoints.

The plan of this paper is as follows. In section 2, the local well-posedness of solution of equation (1.1) is given. In section 3, the L^2-mass concentration of blow-up solutions is proved. In section 4, we construct the blow-up solution with k L^2-mass concentration points. In section 5, some remarks are given.

We conclude this section by giving several notations. We abbreviate $\int_{R^n} .dx$ by $\int .$, $||.||_{L^{2+\frac{4}{n}}(R^n)}$ by $||.||_{L^{2+\frac{4}{n}}}$, and $||.||_{L^2(R^n)}$ by $||.||_{L^2}$.

2 Local Well-posedness

We define a Hilbert space Σ by

$$\Sigma := H^1(R^n) \cap \{u : u|x| \in L^2(R^n)\}$$

with the inner product

$$< u, v >:= \int \nabla u \nabla \overline{v} + u\overline{v} + |x|^2 u\overline{v}, \quad \forall u, v \in \Sigma.$$

The norm of Σ is denoted by $||.||_\Sigma$. Moreover, we define the energy functional E on Σ by

$$E(\psi) := \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\omega^2|x|^2|\psi|^2 + \frac{a}{1+\frac{2}{n}}|\psi|^{2+\frac{4}{n}}, \quad \forall \psi \in \Sigma.$$

According to the Sobolev embedding theorem, the functional E is well defined. From Oh[15], we note that the local well-posedness for the Cauchy problem to (1.1) holds in Σ.

Proposition 1 Let φ in Σ. Then there exists a unique solution of the Cauchy problem (1.1),(1.2) in $C([0,\tau),\Sigma)$ for some $\tau \in (0,+\infty]$ (maximal existence time), and $\tau = +\infty$ or $\tau < +\infty$. If $\tau = +\infty$,solution u is called a global solution. If $\tau < +\infty$, solution u is called blow-up in finite time (also called wave collapse). In addition, u satisfies the following two conservation laws

(i)conservation of mass

$$||u(t)||_{L^2} = ||\varphi||_{L^2}, \qquad t \in [0,\tau). \tag{2.1}$$

(ii)conservation of energy

$$E(u(t)) = E(\varphi), \qquad t \in [0,\tau). \tag{2.2}$$

Consider the nonlinear elliptic equation

$$-\frac{1}{2}\triangle Q + Q = -a|Q|^{\frac{4}{n}}Q, \qquad x \in R^n. \tag{2.3}$$

where $a < 0$. From Weinstein [17] and Kwong [9], we have the following proposition.

Proposition 2 There exists a unique ground state solution $Q(x)$ to (2.3) which is a positive and spherically symmetric solution with exponentially decay at infinity.

Remark 1 It is known that the ground state solution has the minimal L^2-norm, i.e., for any solutions of equation (2.3), $Q'(x)$ different from zero, we have $||Q'||_{L^2} \geq ||Q||_{L^2}$.

3 The L^2-mass concentration of blow-up solution

For a better understanding of the local behavior of blow-up solution to equation (1.1) at blow-up time, this section is devoted to prove that for some blow-up solutions, an L^2−mass concentration phenomenon occurs at blow-up time. In addition, if the initial data φ has a spherical symmetry, an L^2−mass concentration occurs just at the origin 0. Two theorems are established in this section.

Lemma 1[3] (1) Assume that v is the solution of the Cauchy problem (1.3),(1.4) in $C([0,t_0),\Sigma)$ where $t_0 > 0$. Let

$$u(t,x) = \frac{1}{(cos\omega t)^{\frac{n}{2}}} e^{-i\frac{\omega}{2}x^2 tan\omega t} v(\frac{tan\omega t}{\omega}, \frac{x}{cos\omega t}), \tag{3.1}$$

then $u(t,x) \in C([0, \frac{arctan\omega t_0}{\omega}), \Sigma)$ is the solution of Cauchy problem (1.1),(1.2). In particular, if $v \in C([0,+\infty),\Sigma)$(global solution), $u(t) \in C([0, \frac{\pi}{2\omega}), \Sigma)$ is the local solution of Cauchy problem (1.1),(1.2).

(2) Assume that u is the solution of the Cauchy problem (1.1),(1.2) in $C([0,t_0'),\Sigma)$ where $t_0' \in (0, \frac{\pi}{2\omega}]$ .Let

$$v(t,x) = \frac{1}{(1+(\omega t)^2)^{\frac{n}{4}}} e^{i\frac{x^2}{2}\frac{\omega^2 t}{1+\omega^2 t^2}} u(\frac{arctan\omega t}{\omega}, \frac{x}{(1+(\omega t)^2)^{\frac{1}{2}}}), \qquad (3.2)$$

then $v(t,x) \in C([0, \frac{tan\omega t_0'}{\omega}), \Sigma)$ is the solution of the Cauchy problem (1.3),(1.4). In particular, if $u \in C([0, \frac{\pi}{2\omega}), \Sigma)$, $v(t,x) \in C([0,+\infty),\Sigma)$ is the global solution of Cauchy problem (1.3),(1.4).

Remark 2 The transform of lemma 2 is based on the fact that $u(0,x) = v(0,x) = \varphi$. Moreover, the solution obtained by lemma 2 is usually a local solution. The maximal existence time of the local solution for the Cauchy problem relies on the initial data φ.

Based on lemma 1, we get lemma 2.

Lemma 2 Assume that v is the blow-up solution of the Cauchy problem (1.3),(1.4) in $C([0,T),\Sigma)$ for some $T \in (0,+\infty)$ (maximal existence time). Let $u(t,x)$ is defined by (3.1), then

(1) $u(t,x) \in C([0, \frac{arctan\omega T}{\omega}), \Sigma)$ is the blow-up solution of Cauchy problem (1.1),(1.2), where $[0, \frac{arctan\omega T}{\omega})$ is the maximal existence time .

(2) $v(t,x) = \frac{1}{(1+(\omega t)^2)^{\frac{n}{4}}} e^{i\frac{x^2}{2}\frac{\omega^2 t}{1+\omega^2 t^2}} u(\frac{arctan\omega t}{\omega}, \frac{x}{(1+(\omega t)^2)^{\frac{1}{2}}})$, $\quad t \in [0,T)$.(3.2)

Proof: (1) By lemma 1, $u(t,x)$ defined above is the solution of the Cauchy problem (1.1), (1.2) in $C([0, \frac{arctan\omega T}{\omega}), \Sigma)$. Let $[0,\tau)$ be the maximal existence time of u, then $\tau \geq \frac{arctan\omega T}{\omega}$. Let us argue $\tau = \frac{arctan\omega T}{\omega}$ by contradiction. Assume $\tau > \frac{arctan\omega T}{\omega}$, then there are two cases.

(i) $\frac{arctan\omega T}{\omega} < \tau < \frac{\pi}{2\omega}$. Since $u \in C([0,\tau),\Sigma)$, by lemma 1, let $v(t^*,x)$ be defined by (3.2), then $v^*(t,x) \in C([0, \frac{tan\omega\tau}{\omega}), \Sigma)$is the solution of Cauchy problem of (1.3),(1.4). According to the uniqueness of solution of Cauchy problem of equation (1.3), we get $v = v^*$ on $[0, \frac{tan\omega\tau}{\omega})$. But $\frac{tan\omega\tau}{\omega} > T$, it contradicts that $[0,T)$ is the maximal time interval of u.

(ii) $\frac{arctan\omega T}{\omega} < \frac{\pi}{2\omega} < \tau$. Since $u \in C([0, \frac{\tau}{2\omega}))$, according to lemma 1 and the uniqueness of the solution of Cauchy problem of equation (1.3), we directly get that $v(t,x) \in C([0,+\infty),\Sigma)$ is the solution of Cauchy problem (1.3),(1.4), which implies v is a global solution. It contradicts that v blows up in finite time.

Hence $\tau = \frac{arctan\omega T}{\omega}$, which implies $u(t,x)$ blows up in finite time, where $[0, \frac{arctan\omega T}{\omega})$ is the maximal time .

(2)From (1), we get $\tau = \frac{arctan\omega T}{\omega}$, i.,e., $T = \frac{tan\omega\tau}{\omega}$. So we get v^* by (3.2), and $v^* \in C([0, \frac{tan\omega\tau}{\omega}), \Sigma)$. According to the uniqueness of solution of equation (1.3), we have $v = v^*$ on $[0,T)$, where $[0,T)$ is the maximal existence time. This concludes the proof of (2).

For equation (1.3), Merle proved the following two important L^2-mass concentration properties for the blow-up solutions in [10,14].

Lemma 3[10] Assume that v is the blow-up solution of the Cauchy problem (1.3),(1.4) in $C([0,T),\Sigma)$ for some $T \in (0,+\infty)$ (maximal existence time). Then there is an L^2-mass concentration phenomenon for $v(t)$ as $t \to T$: $\exists x(t) \in R^n$, for all $r > 0$,

$$\liminf_{t\to T} ||v(t)||_{L^2(B(x(t),r))} \geq ||Q||_{L^2},$$

where $B(x(t),r) = \{x \in R^n : |x - x(t)| \leq r\}, Q(x)$ is the ground state solution of equation (2.3).

Lemma 4[14] Assume that $n \geq 2$, that φ has a spherical symmetry, and that v is the blow-up solution of the Cauchy problem (1.3),(1.4) in $C([0,T),\Sigma)$ for some $T \in (0,+\infty)$ (maximal existence time). Then for $v(t)$, there is an L^2 mass concentration phenomenon at origin 0 as $t \to T$: for all $r > 0$,

$$\liminf_{t\to T} ||v(t)||_{L^2(B(0,r))} \geq ||Q||_{L^2},$$

where $B(0,r) = \{x \in R^n : |x| \leq r\}, Q(x)$ is the ground state solution of equation (2.3).

With the lemmas above, two theorems are established.

Theorem 1 Assume that v is the blow-up solution of the Cauchy problem (1.3),(1.4) in $C([0,T),\Sigma)$ for some $T \in (0,+\infty)$ (maximal existence time), and that u is the solution of the Cauchy problem (1.1),(1.2) in $C([0,T^*),\Sigma)$ for some $T^* \in (0,+\infty]$ (maximal existence time) . Then

(1) u(t) blows up at time T^*, where $T^* = \frac{arctan\omega T}{\omega} < \frac{\pi}{2\omega}$.

(2) For $u(t)$, an L^2-mass concentration phenomenon occurs as $t \to T^*$: $\exists y(t) \in R^n$, for all $r > 0$,

$$\liminf_{t\to T^*} ||u(t)||_{L^2(B(y(t),r))} \geq ||Q||_{L^2},$$

where $B(y(t),r) = \{x \in R^n : |x - y(t)| \leq r\}, Q(x)$ is the ground state solution of equation (2.3).

Theorem 2 Assume that $n \geq 2$ and initial data φ has a spherical symmetry, that v is the blow-up solution of the Cauchy problem (1.3),(1.4) in $C([0,T),\Sigma)$ for some $T \in (0,+\infty)$ (maximal existence time), and that u is the solution of the Cauchy problem (1.1),(1.2) in $C([0,T^*),\Sigma)$ for some $T^* \in (0,+\infty]$ (maximal existence time) . Then for $u(t)$, an L^2-mass concentration phenomenon occurs at origin 0 as $t \to T^*$: for all $r > 0$,

$$\liminf_{t \to T^*} ||u(t)||_{L^2(B(0,r))} \geq ||Q||_{L^2},$$

where $B(0,r) = \{x \in R^n : |x| \leq r\}, Q(x)$ is the ground state solution of equation (2.3).

In this sense, origin 0 is called an L^2-mass concentration point.

Proof of Theorem 1.

(1) By lemma 2, we define $u^*(t,x)$ by (3.1), then $u^*(t,x) \in C([0, \frac{arctan \omega T}{\omega}), \Sigma)$ is the blow-up solution of Cauchy problem (1.1),(1.2), where $[0, \frac{arctan \omega T}{\omega})$ is the maximal existence time. According to the uniqueness of solution to the Cauchy problem of equation (1.1), $u(t) = u^*(t)$ on $[0, \frac{arctan \omega T}{\omega})$. So $T^* = \frac{arctan \omega T}{\omega} < \frac{\pi}{2\omega}$, which implies that $u(t)$ blows up at time T^*.

(2) By lemma 2,

$$v(t,x) = \frac{1}{(1+(\omega t)^2)^{\frac{n}{4}}} e^{i\frac{x^2}{2}\frac{\omega^2 t}{1+\omega^2 t^2}} u(\frac{arctan \omega t}{\omega}, \frac{x}{(1+(\omega t)^2)^{\frac{1}{2}}}). \quad t \in [0,T) \qquad (3.2)$$

Since $v(t)$ blows up at time T , by lemma 3, there is an L^2-mass concentration phenomenon as $t \to T$ for $v(t)$: $\exists x(t) \in R^n$, for all $r > 0$,

$$\liminf_{t \to T} ||v(t)||_{L^2(B(x(t),r))} \geq ||Q||_{L^2}.$$

By (3.2),

$$\liminf_{t \to T} ||v(t)||_{L^2(B(x(t),r))} = \liminf_{t \to T} (\int_{|x-x(t)| \leq r} |\frac{1}{(1+(\omega t)^2)^{\frac{n}{4}}} u(\frac{arctan \omega t}{\omega}, \frac{x}{(1+(\omega t)^2)^{\frac{1}{2}}})|^2 dx)^{\frac{1}{2}}$$

Let $y = \frac{x}{(1+(\omega t)^2)^{\frac{1}{2}}}$, $t' = \frac{arctan \omega t}{\omega}$, then $t' \in [0, T^*)$.

$$\liminf_{t \to T} ||v(t)||_{L^2(B(x(t),r))} = \liminf_{t' \to T^*} (\int_{|(1+(tan \omega t')^2)^{\frac{1}{2}} y - x(\frac{tan \omega t'}{\omega})| \leq r} |u(t',y)|^2 dy)^{\frac{1}{2}}, \qquad 0 \leq t' < T^*.$$

Put $y(t') = \frac{x(\frac{\tan\omega t}{\omega})}{(1+(\tan\omega t')^2)^{\frac{1}{2}}} \in R^n$, then for all $r > 0$,

$$\liminf_{t\to T} ||v(t)||_{L^2(B(x(t),r))} = \liminf_{t'\to T^*} (\int_{|y-y(t')|\leq \frac{r}{1+(\tan\omega t')^{2\frac{1}{2}}}} |u(t',y)|^2 dy)^{\frac{1}{2}} \geq ||Q||_{L^2}, \quad 0 \leq t' < T^*.$$

$$\liminf_{t'\to T^*} ||u(t')||_{L^2(B(y(t'),r))} = \liminf_{t'\to T^*} (\int_{|y-y(t')|\leq r} |u(t',y)|^2 dy)^{\frac{1}{2}}$$

$$\geq \liminf_{t'\to T^*} (\int_{|y-y(t')|\leq \frac{r}{(1+(\tan\omega t')^2)^{\frac{1}{2}}}} |u(t',y)|^2 dy)^{\frac{1}{2}} \geq ||Q||_{L^2}, \qquad 0 \leq t' < T^*.$$

It doesn't matter to replace t' with t. Hence for $u(t)$ there is an L^2-mass concentration phenomenon as $t \to T^*$: $\exists y(t) \in R^n$, for all $r > 0$,

$$\liminf_{t\to T^*} ||u(t)||_{L^2(B(y(t),r))} \geq ||Q||_{L^2}.$$

This concludes the proof of theorem 1.

In the same way, by lemma 4, theorem 2 can be proved .

4 Construction of the blow-up solution with k L^2-mass concentration points

Since for some spherically symmetric blow-up solutions to (1.1), origin 0 is an L^2-mass concentration point, therefore it is natural to ask if there exists a blow-up solution to (1.1) such that the set of L^2-mass concentration points is different from origin 0, and what behavior it actually has at blow-up time.

In this section, for any k points $x_1, x_2, .., x_k$ in R^n , a blow-up solution is constructed such that an L^2-mass concentration phenomenon occurs at the corresponding k points $cx_1, cx_2, .., cx_k$, where c is a constant that depends on the given k points. In addition, the blow-up solution is sufficiently close to a function at blow-up time.

First of all, we recall an important result obtained by Merle[13] for equation (1.3).

lemma 5[13] Let $x_1, x_2, .., x_k$ be any k points in R^n, $Q_1, Q_2, .., Q_k$ are the spherically symmetric solution of (2.3), then there is a constant $b > 0$ such that as $b_i > b(i = 1, 2, .., k.)$ there exists a blow-up solution u of (1.3) in $C([0, T), \Sigma)$ for some $T \in (0, +\infty)$ (maximal existence time), and u satisfies:

(a)$\lim\limits_{t\to T}||v(t)||_{L^2(B(x_i,r))} = ||Q_i||_{L^2}, \quad \forall r \in A_0, i = 1,2,..,k,$

where $A = \{r > 0 : B(x_i,r) \cap B(x_j,r) = \phi, 1 \leq \neq j \leq k\}$, $B(x_i,r) = \{x \in R^n : |x - x_i| \leq r\}, i = 1,2,..,k,$

(b)$\lim\limits_{t\to T}||v(t)||_{L^2(R^n \setminus \cup_{i=1}^k B(x_i,r))} = 0, \quad \forall r > 0,\ i = 1,2,..,k,$

(c)there exists a constant $\gamma > 0$, such that for $\forall t \in [0,T)$,

$$||v(t) - Q_T(t)||_{L^{2+\frac{4}{n}}} \leq e^{\frac{\gamma}{|T-t|}},$$

where $Q_T(t) = \sum\limits_{i=1}^{k} |(T-t)b_i|^{-\frac{n}{2}} e^{(-\frac{i}{(T-t)b_i^2}) + (\frac{i|x|^2}{2|T-t|})} Q_i(\frac{(x-x_i)}{(T-t)b_i})$.

Remark 3 During the process of proof of lemma 5, F. Merle constructed the blow-up solution $v(t)$ such that $||v(0)||_{L^2}^2 = ||\varphi||_{L^2}^2 = \sum\limits_{i=1}^{k} ||Q_i||_{L^2}^2$.

Now for equation (1.1) which describes Bose-Einstein condensate, the useful results as good as equation (1.3) are proposed and then proved.

Theorem 3 Let $x_1, x_2.., x_k$ be any k points in R^n, $Q_1, Q_2, .., Q_k$ are the spherically symmetric solution of (2.3), then there exists a blow-up solution u of equation (1.1) in $C([0,\tau), \Sigma)$ for some $\tau \in (0, +\infty)$, where $[0,\tau)$ is the maximal existence time, and u satisfies:

(1) The L^2-mass of $u(t)$ concentrates at points $(cos\omega\tau)x_1, (cos\omega\tau)x_2.., (cos\omega\tau)x_k$ in the following sense, for all $r \in A$,

$$\lim_{t\to\tau}||u(t)||_{L^2(B((cos\omega\tau)x_i,r))} = ||Q_i||_{L^2}, \qquad i = 1,2,..,k,$$

where $A = \{r > 0 : B((cos\omega\tau)x_i, r) \cap B((cos\omega\tau)x_j, r) = \phi, 1 \leq i \neq j \leq k\}$, $B((cos\omega\tau)x_i, r) = \{x \in R^n : |x - (cos\omega\tau)x_i| \leq r\}, i = 1,2,..,k.$

(2) For all $r > 0$,

$$\lim_{t\to\tau}||u(t)||_{L^2(R^n \setminus \bigcup\limits_{i=1}^{k} B((cos\omega\tau)x_i,r))} = 0, \qquad i = 1,2,..,k.$$

According to the definition of the ground state solution, we note that

$$\lim_{t\to\tau}||u(t)||_{L^2(B((cos\omega\tau)x_i,r))} = ||Q_i||_{L^2} \geq ||Q||_{L^2},$$

which implies that $(cos\omega\tau)x_i$ is the L^2-mass concentration point, for all $1 \leq i \leq k$.

In order to prove theorem 3, we need to show the following lemma , which is crucial for the proof.

Lemma 6 Let $x_1, x_2, .., x_k$ be any k points in R^n, there exists a blow-up solution u of equation (1.1) in $C([0,\tau),\Sigma)$ for some $\tau \in (0,+\infty)$ (maximal existence time) such that

$$\lim_{t\to\tau} ||u(t)||_{L^2(B((cos\omega t)x_i, rcos\omega t))} = \lim_{t\to\tau} ||u(t)||_{L^2(B((cos\omega\tau)x_i, rcos\omega t))} = ||Q_i||_{L^2},\ r \in A, i = 1,2,..,k. \quad (4.1)$$

Proof of Lemma 6.

Step 1. Get the blow-up solution u.

by Lemma 5, we have $v(t) \in C([0,T),\Sigma)$ that is the blow-up solution of equation (1.3), where $[0,T)$ is the maximal existence time. v has properties (a), (b), (c). According to Lemma 2, let $u(t,x)$ is defined by (3.1), then u is the blow-up solution of equation(1.1), where $\tau = \frac{arctan\omega T}{\omega}$, $[0,\tau)$ is the maximal existence time of u. $u(0) = v(0) = \varphi(x)$.

Step 2. The blow-up solution satisfies (4.1)

Since $cos\omega t > 0$, for all $t \in [0,\tau), \tau < \frac{\pi}{2\omega}$, for all $r \in A$, we have

$$||u(t)||^2_{L^2(B((cos\omega t)x_i, rcos\omega t))} = \int_{|x-(cos\omega t)x_i| \le rcos\omega t} |u(t,x)|^2 dx$$

$$= \int_{|x-(cos\omega t)x_i| \le rcos\omega t} |\frac{1}{(cos\omega t)^{\frac{n}{2}}} v(\frac{tan\omega t}{\omega}, \frac{x}{cos\omega t})|^2 dx = ||v(\frac{tan\omega t}{\omega}, .)||^2_{L^2(B(x_i,r))},\ i = 1,2,..,k.$$

Since it is easy to get $A \subset A_0$, by Lemma 5, we have for all $r \in A$,

$$\lim_{t\to\tau} ||u(t)||_{L^2(B((cos\omega t)x_i, rcos\omega t))} = \lim_{t^*\to T} ||v(t^*)||_{L^2(B(x_i,r))} = ||Q_i||_{L^2}, \quad (4.2)$$

where $t^* = \frac{tan\omega t}{\omega}$ and $t^* \in [0,T)$.

For a fixed i, $\forall t \in [0,\tau)$, we denote by $B_{i\tau t}$ the closed ball in R^n with a fixed center $(cos\omega\tau)x_i$ and changing radius $rcos\omega t$, while by B_{itt} the closed ball in R^n with a moving center $(cos\omega t)x_i$ and changing radius $rcos\omega t$. Namely, $B_{i\tau t} = B((cos\omega\tau)x_i, rcos\omega t) = \{x \in R^n : |x - (cos\omega\tau)x_i| \le rcos\omega t\}$, $B_{itt} = B((cos\omega t)x_i, rcos\omega t) = \{x \in R^n : |x - (cos\omega t)x_i| \le rcos\omega t\}$. Since for all $r \in A$, we have $B_{i\tau t} \cap B_{j\tau t} = \phi (1 \le i \ne j \le k)$. Then it is easy to get that $B_{itt} \cap B_{jtt} = \phi(1 \le i \ne j \le k)$,when t is sufficiently close to τ. And by (2.1),(4.2), and remark 3, we have for all $r \in A$,

$$\lim_{t\to\tau} ||u(t)||^2_{L^2(R^n \setminus \bigcup_{i=1}^k B((cos\omega t)x_i, rcos\omega t))} = ||u(t)||^2_{L^2} - \lim_{t\to\tau} ||u(t)||^2_{L^2(\bigcup_{i=1}^k B((cos\omega t)x_i, rcos\omega t))}$$

$$= \sum_{i=1}^k ||Q_i||^2_{L^2} - \lim_{t\to\tau} \sum_{i=1}^k ||u(t)||^2_{L^2(B((cos\omega t)x_i, rcos\omega t))} = 0, \quad i = 1,2,..,k. \quad (4.3)$$

$\forall t \in [0,\tau)$,we denote $B_{i\tau t} \cap B_{itt}$ by B_{it}, then

$$||u(t)||^2_{L^2(B((cos\omega\tau)x_i,rcos\omega t))} - ||u(t)||^2_{L^2(B((cos\omega t)x_i,rcos\omega t))} = \int_{B_{i\tau t}} |u(t,x)|^2dx - \int_{B_{itt}} |u(t,x)|^2dx$$

$$= \int_{B_{i\tau t}\backslash B_{it}} |u(t,x)|^2dx - \int_{B_{itt}\backslash B_{it}} |u(t,x)|^2dx.$$

To get (4.1), it suffices to show for all $r \in A$,

$$\lim_{t\to\tau} \int_{B_{i\tau t}\backslash B_{it}} |u(t,x)|^2dx = 0, \qquad i = 1,2,..,k. \tag{4.4}$$

$$\lim_{t\to\tau} \int_{B_{itt}\backslash B_{it}} |u(t,x)|^2dx = 0. \qquad i = 1,2,..,k. \tag{4.5}$$

For a fixed i, when t is sufficiently close to τ, we have $x_i cos\omega t \in B_{it} = B_{i\tau t} \cap B_{itt} \neq \phi$, then $x_i cos\omega t \notin B_{i\tau t}\backslash B_{it}$. Furthermore, for any $j \neq i, (1 \leq j \leq k)$, we have $x_j cos\omega t \notin B_{i\tau t}\backslash B_{it}$. Let us argue it by contradiction. Indeed, assume $x_j cos\omega t \in B_{i\tau t}\backslash B_{it} \subset B_{i\tau t}$, then when t is sufficiently close to $\tau, x_j cos\omega t \in B_{itt}$, but $x_j cos\omega t \in B_{jtt}$, which contradicts that $B_{itt} \cap B_{jtt} = \phi$ for $r \in A$. So when t is sufficiently close to τ, for a fixed i, for $r \in A$, we have

$$\{x_i cos\omega t, i = 1,2,..,k.\} \cap (B_{i\tau t}\backslash B_{it}) = \phi,$$

In fact, since i can be arbitrarily chosen, it holds for $i = 1,2,..,k$.

$$(B_{i\tau t}\backslash B_{it}) \subset R^n\backslash\{x_i cos\omega t, i = 1,2,..,k.\}, \qquad i = 1,2,..,k.$$

Then when t is sufficiently close to τ, for sufficiently small $r' \in A(r'$ depends on $r)$, we have

$$(B_{i\tau t}\backslash B_{it}) \subset R^n\backslash \bigcup_{i=1}^{k} B(x_i cos\omega t, r' cos\omega t), \qquad i = 1,2,..,k.$$

By (4.3)

$$0 \leq \lim_{t\to\tau} \int_{B_{i\tau t}\backslash B_{it}} |u(t,x)|^2dx \leq \lim_{t\to\tau} ||u(t)||_{L^2(R^n\backslash \bigcup_{i=1}^{k} B(x_i cos\omega t, r' cos\omega t))} = 0.$$

This concludes (4.4).

In the same way, when t is sufficiently close to τ, we have

$$(B_{itt}\backslash B_{it}) \subset R^n\backslash\{x_i cos\omega t, i = 1,2,..,k.\}, \qquad i = 1,2,..,k.$$

for enough small $r' \in A$,

$$(B_{itt} \backslash B_{it}) \subset R^n \backslash \bigcup_{i=1}^{k} B(x_i cos\omega t, r' cos\omega t), \qquad i = 1, 2, .., k.$$

Then

$$\lim_{t \to \tau} \int_{B_{itt} \backslash B_{it}} |u(t,x)|^2 dx = 0. \tag{4.5}$$

So we get (4.1) by (4.4), (4.5). This concludes the proof of Lemma 6.

With lemma 6, we're in a position to prove theorem 3 now.

Proof of Theorem 3.

By lemma 6, we get the blow-up solution u of equation (1.1) in $C([0,\tau), \Sigma)$ for some $\tau \in (0, +\infty)$ (maximal existence time) such that u satisfies (4.1). We need to show that u has the properties (1), (2).

Step 1. The blow-up solution has the property (1).

For all $r \in A$, by (4.1),

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(B((cos\omega\tau)x_i, r))} \geq \lim_{t \to \tau} ||u(t)||^2_{L^2(B((cos\omega\tau)x_i, r cos\omega t))} = ||Q_i||^2_{L^2}, \qquad i = 1, 2, .., k. \tag{4.6}$$

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(\bigcup_{i=1}^{k} B((cos\omega\tau)x_i, r))} = \lim_{t \to \tau} \sum_{i=1}^{k} ||u(t)||^2_{L^2(B((cos\omega\tau)x_i, r))} \geq \sum_{i=1}^{k} ||Q_i||^2_{L^2}.$$

According to remark 3 and (2.1), for all $r \in A$, we have

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(\bigcup_{i=1}^{k} B((cos\omega\tau)x_i, r))} \leq ||u(t)||^2_{L^2(R^n)} = ||\varphi||^2_{L^2(R^n)} = \sum_{i=1}^{k} ||Q_i||^2_{L^2}.$$

So

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(\bigcup_{i=1}^{k} B((cos\omega\tau)x_i, r))} = \sum_{i=1}^{k} ||Q_i||^2_{L^2}. \tag{4.7}$$

Then for all $r \in A$, we have

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(B((cos\omega\tau)x_i, r))} = ||Q_i||^2_{L^2}, \qquad i = 1, 2, .., k. \tag{4.8}$$

We show (4.8) by contradiction. Assume there is a i such that

$$\lim_{t \to \tau} ||u(t)||^2_{L^2(B((cos\omega\tau)x_i, r))} > ||Q_i||^2_{L^2}$$

then by (4.6),

$$\lim_{t\to\tau}||u(t)||^2_{L^2(\bigcup\limits_{i=1}^{k}B((cos\omega\tau)x_i,r))} = \lim_{t\to\tau}\sum_{i=1}^{k}||u(t)||^2_{L^2(B((cos\omega\tau)x_i,r))} > \sum_{i=1}^{k}||Q_i||^2_{L^2}.$$

It contradicts (4.7). This concludes the proof of (1).

Step 2. The blow-up solution has property (2).

By (4.7), for all $r \in A$,

$$\lim_{t\to\tau}||u(t)||^2_{L^2(R^n\setminus\bigcup_{i=1}^{k}B((cos\omega\tau)x_i,r))} = ||u(t)||^2_{L^2} - \lim_{t\to\tau}||u(t)||^2_{L^2(\bigcup_{i=1}^{k}B((cos\omega t)x_i,r)}$$

$$= \sum_{i=1}^{k}||Q_i||^2_{L^2} - \sum_{i=1}^{k}||Q_i||^2_{L^2} = 0.$$

We can easily check that for all $r > 0$,

$$\lim_{t\to\tau}||u(t)||_{L^2(R^n\setminus\bigcup\limits_{i=1}^{k}B((cos\omega\tau)x_i,r))} = 0. \quad i = 1, 2, .., k.$$

This concludes the proof of theorem 3.

From theorem 3, we get a blow-up solution of with k L^2-mass concentration points, furthermore, we want to know the local behavior of u at blow-up time. This problem is answered by the following theorem.

Theorem 4 For any $x_1, x_2.., x_k$ in R^n, there is a constant $b > 0$ such that for $b_i > b(i = 1, 2, .., k.)$, there exists a blow-up solution u of equation (1.1) in $C([0, \tau), \Sigma)$ for some $\tau \in (0, +\infty)$ (maximal existence time). u not only L^2-concentrates at points $(cos\omega\tau)x_1, (cos\omega\tau)x_2.., (cos\omega\tau)x_k$, but satisfies that there is a constant $\gamma > 0$, such that

$$||u(t) - P_\tau(t)||_{L^{2+\frac{4}{n}}} \leq \frac{1}{(cos\omega t)^{\frac{n}{n+2}}} e^{\frac{\gamma}{|\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega}|}}, \qquad \forall t \in [0, \tau),$$

where

$$P_\tau(t) = \frac{1}{(cos\omega t)^{\frac{n}{2}}} \sum_{i=1}^{k} |(\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega})b_i|^{-\frac{n}{2}} e^{(-\frac{i}{(\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega})b_i^2}) + (\frac{i|x|^2}{2(\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega})(cos\omega t)^2}) + (-i\frac{\omega}{2}x^2 tan\omega t)}$$

$$Q_i(\frac{(x - (cos\omega t)x_i)}{(\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega})b_i cos\omega t}). \qquad (4.9)$$

Remark 4 Theorem 4 implies that in space $L^{2+\frac{4}{n}}$, $u(t) \to P_\tau(t)$, as $t \to \tau$. That's to say, the local behavior of $u(t)$ at blow-up time is sufficiently close to that of $P_\tau(t)$.

Proof of Theorem 4. By lemma 5, for any $x_1, x_2, .., x_k$ in R^n, there is a constant $b > 0$ such that for $b_i > b(i = 1, 2, .., k.)$, , there exists a blow-up solution v of equation (1.3) which satisfies properties (a), (b), (c). From theorem 3, let $u(t,x)$ is defined by (3.1), then $u(t,x) \in C([0,\tau),\Sigma)$ is a blow-up solution of equation (1.1), where $[0,\tau)$ is the maximal existence time. u satisfies properties (1),(2). Let $P_\tau(t)$ be defined by (4.9), consider

$$||u(t)-P_\tau(t)||_{L^{2+\frac{4}{n}}} = ||\frac{1}{(cos\omega t)^{\frac{n}{2}}}[v(\frac{tan\omega t}{\omega}, \frac{x}{cos\omega t})-Q_T(\frac{tan\omega t}{\omega}, \frac{x}{cos\omega t})]||_{2+\frac{4}{n}}$$

$$= \frac{1}{(cos\omega t)^{\frac{n}{n+2}}}||v(\frac{tan\omega t}{\omega}, .) - Q_T(\frac{tan\omega t}{\omega}, .)||_{2+\frac{4}{n}}.$$

As $v(t)$ has property (c), there exists a constant $\gamma > 0$, such that

$$||v(t) - Q_T(t)||_{L^{2+\frac{4}{n}}} \le e^{\frac{\gamma}{|T-t|}}, \qquad \forall t \in [0,T).$$

Note that $\tau = \frac{arctan\omega T}{\omega}$, i.e., $\frac{tan\omega\tau}{\omega} = T$ Then there exists a constant $\gamma > 0$, such that

$$||u(t) - P_\tau(t)||_{L^{2+\frac{4}{n}}} \le \frac{1}{(cos\omega t)^{\frac{n}{n+2}}} e^{\frac{\gamma}{|\frac{tan\omega\tau}{\omega} - \frac{tan\omega t}{\omega}|}}, \qquad \forall t \in [0,\tau).$$

This concludes the proof of theorem 4.

5 Some Remarks

From comparison between lemma 5 and theorem 3, we can find, since equation (1.1) hasn't the dilation invariance, harmonic potential $|x|^2$ exerts a great influence on equation (1.1). In fact, for any k points in R^n, Merle[13] has constructed a blow-up solution such that an L^2-mass concentration phenomenon occurs just at the k points and the set of blow-up points is the k points . The definition of blow-up solution is following:

Definition: $x_0 (x_0 \in R^n)$ is called a blow-up point of solution $u(t)$ in space $L^{2+\frac{4}{n}}$ (or $H^1(R^n)$), if:

$$\lim_{t\to\tau} ||u(t)||_{L^{2+\frac{4}{n}}(B(x_0,r))} = +\infty, \qquad \forall r > 0.$$

$$or \quad \lim_{t\to\tau} ||u(t)||_{H^1(B(x_0,r))} = +\infty, \qquad \forall r > 0.$$

In this paper, due to the influence of harmonic potential, L^2-mass of the blow-up solution constructed by theorem 3 concentrates not at the original k points but at the corresponding k other points whose point-vector lengths are reduced. However, whether or not the corresponding k points constitute the set of blow-up points of the blow-up solution is the problem that we're concerned about. Since we can't turn the limit of integral on neighborhoods of moving point into the limit of integral on neighborhoods of a fixed point by technically treatment, we will make further study on these problems in the work later.

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